Tangential Navier-Stokes equations on evolving surfaces: Analysis and simulations

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Joint work with Philip Brandner, Paul Schwering (RWTH), Maxim Olshanskii (Houston)

Outline

- Surface Navier-Stokes equations
- Well-posedness of tangential surface NS equations.
- Discretization method

Applications/related work:
Fluid deformable surfaces: talk of A. Voigt (tuesday June 6th).
We consider **only** viscous surface flows, **no** bending energies, **no** multiphase.

Modeling of fluidic **evolving** surfaces

[P1] Hu, Zhang, and E, *Continuum theory of a moving membrane*, Phys. Rev. E (2007)

[P2] Jankuhn, Olshanskii, AR,
Incompressible fluid problems on embedded surfaces: Modeling and variational formulations, IFB (2018)

[P3] Koba, Liu, Giga,
Energetic variational approaches for incompressible fluid systems on an evolving surface,
Quart. Appl. Math. (2017)

[P4] Miura,
On singular limit equations for incompressible fluids in moving thin domains, Quart. Appl. Math. (2017)

[P5] Nitschke, Reuther, Voigt,
Hydrodynamic interactions in polar liquid crystals on evolving surfaces,
Phys. Rev. Fluids (2019)

Modeling aspects

A. Physical principles

- Volume mass/momentum conservation + thin film limit [P4], [P5].
- Surface mass/momentum conservation [P1], [P2].
- Energetic principles [P3].

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- Local coordinate system: curvilinear coordinates.
- Global Cartesian coordinate system.

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- Energetic principles [P3].

B. Mathematical representation

- Local coordinate system: curvilinear coordinates.
- Global Cartesian coordinate system.

$\rightsquigarrow$ resulting equations look (very) different.

Overview paper: P. Brander, AR, P. Schwering,
On derivations of evolving surface Navier–Stokes equations. IFB 2022

$\rightsquigarrow$ resulting equations are the same:
The evolving surface Navier-Stokes equations.

The surface NS equations

Surface differential operators: ∇_Γ, div_Γ. Normal vector: $\mathbf{n}$.

Velocity $\mathbf{u} = \mathbf{u}_T + \mathbf{u}_N = \mathbf{u}_T + u_N\mathbf{n}$, surface pressure p. Density $\rho := 1$.

Material derivative: $\overset{\bullet}{\mathbf{u}} = \partial_t \mathbf{u}^e + \mathbf{u} \cdot \nabla \mathbf{u}^e$ (along space-time surface).

Rate of strain tensor $\mathbf{E}(\mathbf{u}) = \frac{1}{2}(\nabla_\Gamma \mathbf{u} + \nabla_\Gamma \mathbf{u}^T)$, κ: mean curvature.

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Full surface NSE (for $\mathbf{f} = 0$) on $\Gamma(t)$: $\mathbf{u}$, p such that

$$\begin{aligned} \overset{\bullet}{\mathbf{u}} - 2\mu\, \mathrm{div}_\Gamma \mathbf{E}(\mathbf{u}) + p\,\kappa\mathbf{n} + \nabla_\Gamma p &= 0 \\ \mathrm{div}_\Gamma \mathbf{u} &= 0 \end{aligned}$$

Note: $\mathbf{u}_N = u_N\mathbf{n}$ defines $\Gamma(t)$ (free surface problem).

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Well-posedness of this free surface Navier-Stokes system: open problem
Collaboration with H. Abels (work in progress)

Extensive analysis for Navier-Stokes on stationary surfaces/manifolds:
[Arnold], [Ebin-Marsden],...

Tangential surface NS

Huge simplification: $\mathbf{u}_N =: \mathbf{w}_N = w_N \mathbf{n}$ given (i.e. $\Gamma(t)$ given).

Tangential projection of full system, $\mathbf{P} := \mathbf{I} - \mathbf{n}\mathbf{n}^T$, $\mathbf{H} :=: \nabla_\Gamma \mathbf{n}$.

Material derivative along normal flow $\mathbf{w}_N$: ∂°.

Tangential surface NS (TSNSE): $\mathbf{u}_T$, p such that on $\Gamma(t)$:

$$\mathbf{P}\partial^\circ \mathbf{u}_T + w_N \mathbf{H}\mathbf{u}_T + (\nabla_\Gamma \mathbf{u}_T)\mathbf{u}_T - 2\mu \mathbf{P}\,\mathrm{div}_\Gamma \mathbf{E}(\mathbf{u}_T) + \nabla_\Gamma p = \mathbf{f}$$

$$\mathrm{div}_\Gamma \mathbf{u}_T = g$$

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\end{aligned}
$$

Note: $\mathbf{u}_T$ tangential to $\Gamma(t)$.
In the remainder: space-time variational formulation of TSNSE ($g = 0$).

Outline of analysis

TSNSE unknowns: tangential velocity $\mathbf{u}_T$ and pressure p.

Key points of analysis:

- Smoothness assumptions on space-time surface $\mathcal{S} = \cup_{t\in[0,T]}\Gamma(t)$.
- Suitable evolving Hilbert spaces [Alphonse, Elliott, Stinner, 2015].
- Relate different "material derivatives".
- Uniform in time key estimates (e.g., Korn inequality).
- Mimic the standard approach for tensor space-time cylinder [Temam].

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Smoothness assumptions:

$$\Gamma_0 = \Gamma(0) \in C^3. \quad \text{Geometric transport by } \mathbf{w} \in C^3([0,T]\times\mathbb{R}^3,\mathbb{R}^3)$$

Normal flow map $\Phi_t^n : \Gamma_0 \to \Gamma(t)$ based on $\mathbf{w}_N$.

Evolving Hilbert spaces

We need pushforward map: $\mathbf{v}$ on $\Gamma_0 \to \phi_t \mathbf{v}$ on $\Gamma(t)$.

Surface Piola transform pushforward

$$J := \det(D\Phi_t^n), \quad \mathbf{A} := J^{-1} D\Phi_t^n : T\Gamma_0 \to T\Gamma(t)$$

$$(\phi_t \mathbf{v})(\mathbf{x}) := \mathbf{A}(t, \mathbf{z})\mathbf{v}(z), \quad \mathbf{z} \in \Gamma_0, \; \mathbf{x} = \Phi_t^n(\mathbf{z}) \in \Gamma(t)$$

Property $\mathrm{div}_{\Gamma_0} \mathbf{v} = 0 \;\Leftrightarrow\; \mathrm{div}_{\Gamma(t)} (\phi_t \mathbf{v}) = 0$.

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Property $\mathrm{div}_{\Gamma_0} \mathbf{v} = 0 \ \Leftrightarrow\ \mathrm{div}_{\Gamma(t)}(\phi_t \mathbf{v}) = 0$.

Spaces of tangential velocity fields on $\Gamma(t)$:

$$H^1(t) := \{\mathbf{v} \in H^1(\Gamma(t))^3 \mid \mathbf{v} \cdot \mathbf{n} = 0 \ \text{ a.e. on } \ \Gamma(t)\}$$

$$V(t) := \{\, \mathbf{v} \in H^1(t) \mid \mathrm{div}_\Gamma \mathbf{v} = 0 \ \text{ a.e. on } \ \Gamma(t)\,\}$$

$$\mathcal{V}(t) := \{\, \mathbf{v} \in C^1(\Gamma(t))^3 \mid \mathbf{v} \cdot \mathbf{n} = 0,\ \mathrm{div}_\Gamma \mathbf{v} = 0 \ \text{ on } \ \Gamma(t)\,\}$$

$$H(t) := \overline{\mathcal{V}(t)}^{\|\cdot\|_{L^2(\Gamma(t))}}$$

$\{V(t), \phi_t\}$, $\{H(t), \phi_t\}$ are compatible pairs [Alphonse et al.]

Induce evolving spaces (generalization of standard Bochner spaces):

$$L^2_V = \{ (t, \bar{\mathbf{v}}(t)) \in \bigcup_{t \in [0,T]} \{t\} \times V(t) \mid \\ \phi_{-(\cdot)} \bar{\mathbf{v}}(\cdot) \in L^2(0, T; V(0)) \}$$

Similarly: $L^2_{H^1}$, L^2_H, $L^2_{V'}$, $L^2_{H^{-1}}$.

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Similarly: $L^2_{H^1}$, L^2_H, $L^2_{V'}$, $L^2_{H^{-1}}$.

Natural topology. Usual nice properties:

- Hilbert spaces (canonical inner products)
- Density of smooth functions
- $L^2_V \hookrightarrow L^2_H \hookrightarrow L^2_{V'}$ Gelfand triple, dense and compact embedding.

Material derivatives

Recall:

Full material derivative $\dot{\mathbf{v}} = \partial_t \mathbf{v}^e + \mathbf{u} \cdot \nabla \mathbf{v}^e$ ($\mathbf{u}$: solution of full NS).

Normal material derivative $\partial^\circ \mathbf{v} = \partial_t \mathbf{v}^e + \mathbf{w}_N \cdot \nabla \mathbf{v}^e$ ($\mathbf{w}_N = \mathbf{u}_N$ given).

Another Lagrangian derivative (used in [Alphonse et al.])

$$\partial^* \mathbf{v}(t) := \phi_t \left(\frac{d}{dt} \phi_{-t} \mathbf{v}(t) \right) \quad (\text{“}\phi_t\text{-derivative”})$$

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Lemma

$$\partial^\circ \mathbf{v} = \partial^* \mathbf{v} - \mathbf{C}\mathbf{v}, \quad \mathbf{C} := \mathbf{A}(\partial^\circ \mathbf{A}^{-1})$$

$\mathbf{A}$ depends (only) on Φ_t^n.

Hence $\partial^\circ \mathbf{v} \sim \partial^* \mathbf{v}$.

Variational formulation

Solution space:

$$\mathbf{W}(V, V') = \{ \mathbf{v} \in L^2_V \mid \partial^\circ \mathbf{v} \in L^2_{V'} \}.$$

Use $\partial^\circ \sim \partial^*$ and [Alphonse et al.] $\Rightarrow$:
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Notation: $\int_0^T \int_{\Gamma(t)} \mathbf{u} \cdot \mathbf{v} \, ds \, dt =: (\mathbf{u}, \mathbf{v})_{\mathcal{S}}$.

$$a(\mathbf{u}, \mathbf{v}) := 2\mu(\mathbf{E}(\mathbf{u}), \mathbf{E}(\mathbf{v}))_S, \quad c(\mathbf{u}, \tilde{\mathbf{u}}, \mathbf{v}) := ((\nabla_\Gamma \mathbf{u})\tilde{\mathbf{u}}, \mathbf{v})_S,$$
$$\ell(\mathbf{u}, \mathbf{v}) := (w_N \mathbf{H}\mathbf{u}, \mathbf{v})_S$$

Variational formulation of TSNSE

Given $\mathbf{f} = \mathbf{f}_T \in L^2(\mathcal{S})^3$, $\mathbf{u}_0 \in H(0)$, find $\mathbf{u}_T \in \mathbf{W}(V, V')$ such that $\mathbf{u}_T(0) = \mathbf{u}_0$ and

$$\langle \partial^\circ \mathbf{u}_T, \mathbf{v} \rangle + \ell(\mathbf{u}_T, \mathbf{v}) + c(\mathbf{u}_T, \mathbf{u}_T, \mathbf{v}) + a(\mathbf{u}_T, \mathbf{v}) = (\mathbf{f}, \mathbf{v})_S \quad \text{for all } \ \mathbf{v} \in L^2_V.$$

Analysis of well-posedness

Differences to $\otimes$ case: "other" spaces, ∂° instead of $\frac{\partial}{\partial t}$, $\ell(\cdot,\cdot)$ term.

Uniform estimates (Korn, inf-sup, interpolation/Ladyzhenskaya)

$$\|\mathbf{v}\|_{H^1(\Gamma(t))} \leq c\Big(\|\mathbf{v}\|_{L^2(\Gamma(t))} + \|\mathbf{E}(\mathbf{v})\|_{L^2(\Gamma(t))}\Big) \quad \text{for all } \mathbf{v} \in H^1(t)$$

$$\sup_{0\neq \mathbf{v}\in H^1(t)} \frac{\int_{\Gamma(t)} p \,\mathrm{div}_\Gamma \mathbf{v}\, ds}{\|\mathbf{v}\|_{H^1(t)}} \geq c\|p\|_{L^2(t)}, \quad \forall\, p \in L^2(\Gamma(t)), \ \int_{\Gamma(t)} p = 0$$

$$\|\mathbf{v}\|_{L^4(\Gamma(t))} \leq c\|\mathbf{v}\|_{L^2(\Gamma(t))}^{\frac{1}{2}}\|\mathbf{v}\|_{H^1(\Gamma(t))}^{\frac{1}{2}}, \quad \mathbf{v} \in H^1(\Gamma(t)) \quad \text{(2D surface)}$$

with $c > 0$ independent of $t \in [0, T]$.

Follow the approach of [Temam] for $\otimes$ case

I. Feado-Galerkin approximation:

$$\mathbf{u}_m := \sum_{i=1}^{m} g_{i,m}(t)\widetilde{\boldsymbol{\psi}}_i, \quad \text{with countable basis } \{\widetilde{\boldsymbol{\psi}}_i\} \text{ of } V(t).$$

$\mathbf{P}\partial^\circ\widetilde{\boldsymbol{\psi}}_i = \partial^*\widetilde{\boldsymbol{\psi}}_i - \mathbf{C}\widetilde{\boldsymbol{\psi}}_i = -\mathbf{C}\widetilde{\boldsymbol{\psi}}_i$.
Existence of $g_{i,m}$ solution via ODE theory.

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Existence of $g_{i,m}$ solution via ODE theory.
II. A-priori bounds for $\mathbf{u}_m$ (use uniform estimates), e.g.

$$\max_{0\leq t\leq T} \|\mathbf{u}_m\|_{L^2(\Gamma(t))} + \|\mathbf{E}(\mathbf{u}_m)\|_{L^2(\mathcal{S})} \lesssim \|\mathbf{f}\|_{L^2(\mathcal{S})} + \|\mathbf{u}_0\|_{L^2(\Gamma_0)}$$

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III. Existence of solution $\mathbf{u}$: boundedness and compactness arguments.

IV. Uniqueness of solution $\mathbf{u}$: uniform interpolation estimate, uniform Korn inequality and Gronwall argument.

Mixed weak formulation of TSNSE

Given $\mathbf{f} = \mathbf{f}_T \in L^2(\mathcal{S})^3$, $\mathbf{u}_0 \in H(0)$, find $\mathbf{u}_T \in \mathbf{W}(H^1, H^{-1})$ with $\mathbf{u}_T(0) = \mathbf{u}_0$, $p \in L^2(\mathcal{S})$ with $\int_{\Gamma(t)} p\, ds = 0$ a.e. $t \in [0, T]$, such that

$$\begin{aligned} \langle \partial^\circ \mathbf{u}_T, \mathbf{v} \rangle + \ell(\mathbf{u}_T, \mathbf{v}) + c(\mathbf{u}_T, \mathbf{u}_T, \mathbf{v}) & \\ + a(\mathbf{u}_T, \mathbf{v}) + (p, \operatorname{div}_\Gamma \mathbf{v})_{\mathcal{S}} &= (\mathbf{f}, \mathbf{v})_S \quad \text{for all } \mathbf{v} \in L^2_{H^1} \\ (q, \operatorname{div}_\Gamma \mathbf{u}_T)_{\mathcal{S}} &= 0 \quad \text{for all } q \in L^2(\mathcal{S}). \end{aligned}$$

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\end{aligned}
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Theorem

This problem is well-posed.

Proof essentially the same as in $\otimes$ case, based on uniform inf-sup property and closed range theorem.

Note: global result and no "smallness" condition.

Discretization methods

Comparison of methods for stationary surfaces:

[Brandner et al., *Finite element discretization methods for velocity-pressure and stream function formulations of surface Stokes equations*, SISC 2022]

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cf. Talk of A. Voigt,

Alternative for evolving surfaces with topological singularities (droplet pinch-off): TraceFEM.

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Error analysis for surface (Navier-)Stokes:
available only for TraceFEM and Stokes on stationary surface.

Recent work: error analysis of TraceFEM for TSNSE.

TraceFEM for TSNSE

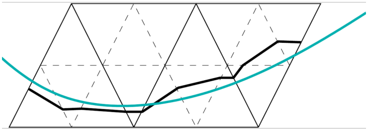
Based on TraceFEM (available in NGSolve). Key ingredients:

- Level set representation:

$$\Gamma(t) = \{\mathbf{x} \in \mathbb{R}^3 \, : \, \phi(t, \mathbf{x}) = 0\}$$

- $\phi_h \approx \phi$, $\Gamma_h(t) = \{\mathbf{x} \in \mathbb{R}^3 \, : \, \phi_h(t, \mathbf{x}) = 0\}$ given. $\Gamma_h^n := \Gamma_h(t_n)$.
  In practice: parametric TraceFEM [Lehrenfeld].
- Outer shape regular triangulation $\mathcal{T}_h$. Narrow bands

$$\omega_\Gamma^n := \bigcup \left\{ \overline{K} \in \mathcal{T}_h \, : \, K \cap \Gamma_h^n \neq \emptyset \right\}, \quad \mathcal{O}_\delta(\Gamma_h^n) \supset \omega_\Gamma^n.$$



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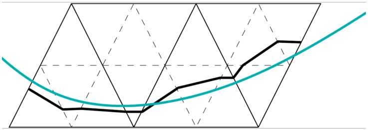

- standard FE spaces $\mathbf{U}_h$, Q_h on $\mathcal{T}_h$ (e.g. Hood-Taylor)

$$\mathbf{U}_h^n := \{ \, \mathbf{v}|_{\mathcal{O}_\delta(\Gamma_h^n)} \mid \mathbf{v} \in \mathbf{U}_h \, \}, \quad Q_h^n := \{ \, q|_{\omega_\Gamma^n} \mid q \in Q_h \, \}.$$

Discretization method: one time step BDF1

For given $\mathbf{u}_h^{n-1} \in \mathbf{U}_h^{n-1}$ find $\mathbf{u}_h^n \in \mathbf{U}_h^n$, $p_h^n \in Q_h^n$, such that $\forall\ \mathbf{v}_h \in \mathbf{U}_h^n$:

$$\int_{\Gamma_h^n} \left(\frac{\mathbf{u}_h^n - \mathbf{u}_h^{n-1}}{\Delta t} + \mathbf{w}_N \cdot \nabla \mathbf{P}_h \mathbf{u}_h^n + w_N \mathbf{H}_h \mathbf{u}_h^n \right) \cdot \mathbf{P}_h \mathbf{v}_h \, ds_h$$

$$+ \frac{1}{2} \int_{\Gamma_h^n} (\mathbf{u}_h^{n-1} \cdot \nabla_{\Gamma_h} \mathbf{P}_h \mathbf{u}_h^n) \cdot \mathbf{v}_h - (\mathbf{u}_h^{n-1} \cdot \nabla_{\Gamma_h} \mathbf{P}_h \mathbf{v}_h) \cdot \mathbf{u}_h^n \, ds_h$$

$$+ 2\mu \int_{\Gamma_h^n} \mathbf{E}_h(\mathbf{P}_h \mathbf{u}_h^n) : \mathbf{E}_h(\mathbf{P}_h \mathbf{v}_h) \, ds_h + \underbrace{\tau \int_{\Gamma_h^n} (\tilde{\mathbf{n}}_h \cdot \mathbf{u}_h^n)(\tilde{\mathbf{n}}_h \cdot \mathbf{v}_h) \, ds_h}_{\text{penalty for } \mathbf{n} \cdot \mathbf{u}^n = 0}$$

$$+ \int_{\Gamma_h^n} \nabla_{\Gamma_h} p_h^n \cdot \mathbf{v}_h \, ds_h + \underbrace{\rho_u \int_{\mathcal{O}_\delta \Gamma_h^n)} (\mathbf{n}_h \cdot \nabla \mathbf{u}_h^n)(\mathbf{n}_h \cdot \nabla \mathbf{v}_h) \, dx}_{\text{stabilization and constant extension}} = \int_{\Gamma_h^n} \mathbf{f} \cdot \mathbf{v}_h \, ds_h$$

Discretization method: one time step BDF1

For given $\mathbf{u}_h^{n-1} \in \mathbf{U}_h^{n-1}$ find $\mathbf{u}_h^n \in \mathbf{U}_h^n$, $p_h^n \in Q_h^n$, such that $\forall\ \mathbf{v}_h \in \mathbf{U}_h^n$:

$$\int_{\Gamma_h^n} \left(\frac{\mathbf{u}_h^n - \mathbf{u}_h^{n-1}}{\Delta t} + \mathbf{w}_N \cdot \nabla \mathbf{P}_h \mathbf{u}_h^n + w_N \mathbf{H}_h \mathbf{u}_h^n \right) \cdot \mathbf{P}_h \mathbf{v}_h \, ds_h$$

$$+ \tfrac{1}{2} \int_{\Gamma_h^n} (\mathbf{u}_h^{n-1} \cdot \nabla_{\Gamma_h} \mathbf{P}_h \mathbf{u}_h^n) \cdot \mathbf{v}_h - (\mathbf{u}_h^{n-1} \cdot \nabla_{\Gamma_h} \mathbf{P}_h \mathbf{v}_h) \cdot \mathbf{u}_h^n \, ds_h$$

$$+ 2\mu \int_{\Gamma_h^n} \mathbf{E}_h(\mathbf{P}_h \mathbf{u}_h^n) : \mathbf{E}_h(\mathbf{P}_h \mathbf{v}_h) \, ds_h + \underbrace{\tau \int_{\Gamma_h^n} (\tilde{\mathbf{n}}_h \cdot \mathbf{u}_h^n)(\tilde{\mathbf{n}}_h \cdot \mathbf{v}_h) \, ds_h}_{\text{penalty for } \mathbf{n} \cdot \mathbf{u}^n = 0}$$

$$+ \int_{\Gamma_h^n} \nabla_{\Gamma_h} p_h^n \cdot \mathbf{v}_h \, ds_h + \underbrace{\rho_u \int_{\mathcal{O}_\delta \Gamma_h^n)} (\mathbf{n}_h \cdot \nabla \mathbf{u}_h^n)(\mathbf{n}_h \cdot \nabla \mathbf{v}_h) \, dx}_{\text{stabilization and constant extension}} = \int_{\Gamma_h^n} \mathbf{f} \cdot \mathbf{v}_h \, ds_h$$

$$- \int_{\Gamma_h^n} \nabla_{\Gamma_h} q_h \cdot \mathbf{u}_h^n \, ds_h + \underbrace{\rho_p \int_{\omega_\Gamma^n} (\mathbf{n}_h \cdot \nabla p_h^n)(\mathbf{n}_h \cdot \nabla q_h) \, dx}_{\text{pressure stabilization}} = 0 \quad \forall\ q_h \in Q_h^n.$$

Discretization error analysis

Scaling of penalty and stability parameters based on analysis:

$$\tau \sim h^{-2}, \quad \rho_u \sim h^{-1}, \quad \rho_p \sim h$$

Finite element spaces:
For geometry: ϕ_h piecewise P_q.
For $(\mathbf{u}, p)$: Hood-Taylor $\mathbf{P}_{m+1} - P_m, \quad m \geq 1$.

A natural energy norm $|||\cdot|||$ is used. Weak CFL-type condition $\Delta t \lesssim h$.

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Velocity error estimate

$$\|\mathbf{u}(t_n,\cdot) - \mathbf{u}_h^n\|^2_{L^2(\Gamma_h^n)} + \Delta t \sum_{k=1}^{n} |||\mathbf{u}(t_k,\cdot) - \mathbf{u}_h^k|||^2 \leq c(\Delta t^2 + h^{2(m+1)} + h^{2q})$$

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Velocity error estimate

$$\|\mathbf{u}(t_n,\cdot) - \mathbf{u}_h^n\|_{L^2(\Gamma_h^n)}^2 + \Delta t \sum_{k=1}^{n} |\!|\!|\mathbf{u}(t_k,\cdot) - \mathbf{u}_h^k|\!|\!|^2 \leq c(\Delta t^2 + h^{2(m+1)} + h^{2q})$$

- Analysis includes geometry approximation
- The error bound is optimal
- Also an optimal bound for the pressure error is derived
- Analysis can be extended to BDF2 ($\Delta t^2 \rightsquigarrow \Delta t^4$)

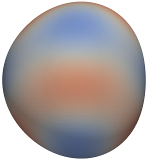
Numerical experiment

Tangential surface NS (TSNSE); $w_N := u_N$ given

$$\mathbf{P}\partial^\circ \mathbf{u}_T + w_N \mathbf{H}\mathbf{u}_T + (\nabla_\Gamma \mathbf{u}_T)\mathbf{u}_T - 2\mu \mathbf{P}\,\mathrm{div}_\Gamma \mathbf{E}(\mathbf{u}_T) + \nabla_\Gamma p = \mathbf{f}$$

$$\mathrm{div}_\Gamma \mathbf{u}_T = -w_N \kappa$$

with $\mathbf{f} = 2\mu \mathbf{P}\,\mathrm{div}_\Gamma(w_N \mathbf{H}) + \frac{1}{2}\nabla_\Gamma w_N^2$.



Flow is completely "geometry driven"

Remarks

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References:

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