Numerical computation for 4th order total variation flow

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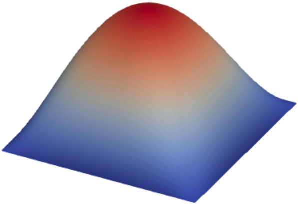
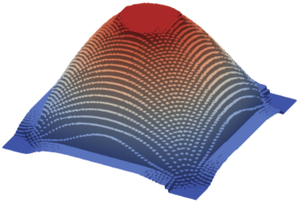
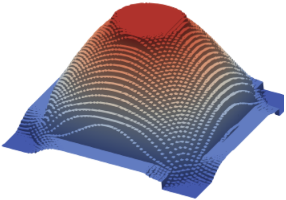




4th order TV flow

$$u_t = -\Delta\left(\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)\right) \text{ in } \mathbb{T}^n.$$

- This is characterized as $u_t \in -\partial_{H^{-1}_{\mathrm{av}}(\mathbb{T}^n)} \mathrm{TV}(u)$.
- The inverse Laplacian on torus: $(-\Delta_{\mathrm{av}})^{-1} : H^{-1}_{\mathrm{av}}(\mathbb{T}^n) \to H^{1}_{\mathrm{av}}(\mathbb{T}^n)$.
  - This gives $\|u\|_{H^{-1}_{\mathrm{av}}(\mathbb{T}^n)} = \|(-\Delta_{\mathrm{av}})^{-1}u\|_{H^{1}_{\mathrm{av}}(\mathbb{T}^n)} = \|\nabla(-\Delta_{\mathrm{av}})^{-1}u\|_{L^2(\mathbb{T}^n)}$
- We apply the backward Euler method:

$$\frac{u^{k+1} - u^k}{\tau} \in -\partial_{H^{-1}_{\mathrm{av}}(\mathbb{T}^n)} \mathrm{TV}(u^{k+1}).$$

- Introduce constraint condition $d = Du$:

$$u^{k+1} = \operatorname*{argmin}_{u \in H^{-1}_{\mathrm{av}}(\mathbb{T}^n)} \left\{ \int_{\mathbb{T}^n} |d| + \frac{1}{2\tau}\|u - u^k\|_{H^{-1}_{\mathrm{av}}(\mathbb{T}^n)} : d = Du \right\}.$$

Discretization and the split Bregman framework

- u and d are approximated by the piecewise constant functions.
- Let $\nabla_{h,\mathrm{av}}$ be the discretized gradient and $(-\Delta_{h,\mathrm{av}})^{-1}$ be the discretized inverse Laplacian.

$$(\boldsymbol{u}^{k+1}, \boldsymbol{d}^{k+1}) = \operatorname*{argmin}_{\boldsymbol{u},\boldsymbol{d}} \left\{ \|\boldsymbol{d}\|_1 + \frac{\tau^{-1}}{2} \|\nabla_{h,\mathrm{av}}(-\Delta_{h,\mathrm{av}})^{-1}(\boldsymbol{u} - \boldsymbol{u}^k)\|_2^2 + \frac{\mu}{2}\|\boldsymbol{d} - \nabla_{h,\mathrm{av}}\boldsymbol{u}\|_2^2 \right\}.$$

Split Bregman framework

$$\begin{cases} \boldsymbol{u}^{k,j+1} &= \operatorname*{argmin}_{\boldsymbol{u}} \left\{ \frac{\tau^{-1}}{2} \|\nabla_{h,\mathrm{av}}(-\Delta_{h,\mathrm{av}})^{-1}(\boldsymbol{u} - \boldsymbol{u}^k)\|_2^2 + \frac{\mu}{2}\|\boldsymbol{d}^{k,j} - \nabla_{h,\mathrm{av}}\boldsymbol{u} - \boldsymbol{b}^{k,j}\|_2^2 \right\}, \\ \boldsymbol{d}^{k,j+1} &= \operatorname*{argmin}_{\boldsymbol{d}} \left\{ \|\boldsymbol{d}\|_1 + \frac{\mu}{2}\|\boldsymbol{d} - \nabla_{h,\mathrm{av}}\boldsymbol{u}^{k,j+1} - \boldsymbol{b}^{k,j}\|_2^2 \right\}, \\ \boldsymbol{b}^{k,j+1} &= -\left(\boldsymbol{d}^{k,j+1} - \nabla_{h,\mathrm{av}}\boldsymbol{u}^{k,j+1} - \boldsymbol{b}^{k,j}\right). \end{cases}$$

- $\boldsymbol{u}^{k+1}$ is given as the limit: $\boldsymbol{u}^{k+1} = \lim_{j\to\infty} \boldsymbol{u}^{k,j}$.

Numerical example: 1D case

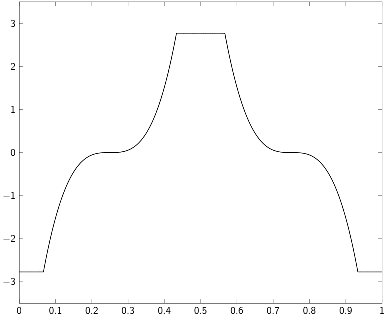


Figure: 4th order TV flow

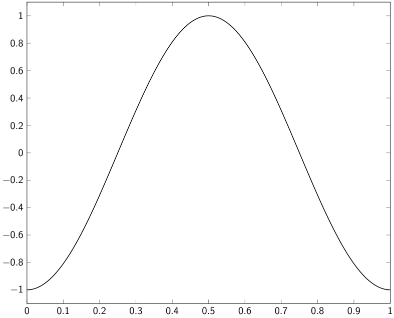
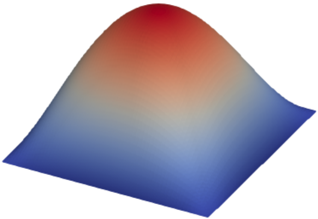
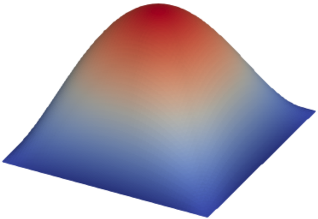
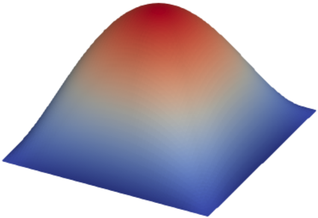


Figure: Spohn's model for crystal growth

Numerical example: 2D case







- Upper Left: Anisotropic TV flow
- Upper Right: Isotropic TV flow
- Lower Left: Spohn's model