Stability of volume-preserving mean curvature flow & optimal convergence rates for the nonlocal Allen–Cahn equation

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Volume-preserving mean curvature flow

Find evolving surface $\Sigma(t)$ such that

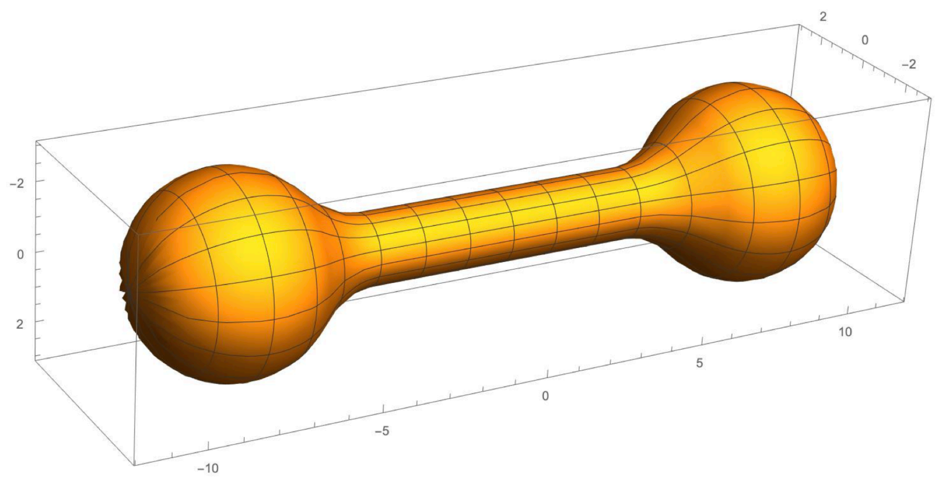
$$V = -H + \lambda \quad \text{on } \Sigma(t),$$

where

$$V = V(x,t) = \text{normal velocity},$$

$$H = H(x,t) = \text{mean curvature} = \sum_{i=1}^{d-1} \kappa_i,$$

$$\lambda = \lambda(t) := \frac{1}{\mathcal{H}^{d-1}(\Sigma(t))} \int_{\Sigma(t)} H \, \mathrm{d}\mathcal{H}^{d-1}.$$



Gradient-flow structure

Energy:	$E[\Sigma] = \mathcal{H}^{d-1}(\Sigma)$,
State space:	$\mathcal{M} = \big\{\Sigma = \partial\Omega \subset \mathbb{R}^d \,\big|\, \lvert\Omega\rvert = m\big\}$,
Tangent space:	$T_\Sigma\mathcal{M} \cong \big\{V : \Sigma \to \mathbb{R} \,\big|\, \int_\Sigma V \,\mathrm{d}\mathcal{H}^{d-1} = 0\big\}$,
Metric tensor:	$(V, W)_\Sigma = \int_\Sigma V W \,\mathrm{d}\mathcal{H}^{d-1}$.

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Metric tensor: $(V, W)_\Sigma = \int_\Sigma VW \,\mathrm{d}\mathcal{H}^{d-1}$.

Indeed, if $\Sigma(t)$ is a smooth solution of $V = -H + \lambda$, then

$$\frac{\mathrm{d}}{\mathrm{d}t} E[\Sigma(t)] = \int_{\Sigma(t)} V(H - \lambda(t)) \,\mathrm{d}\mathcal{H}^{d-1} = -\int_{\Sigma(t)} V^2 \,\mathrm{d}\mathcal{H}^{d-1}$$

and

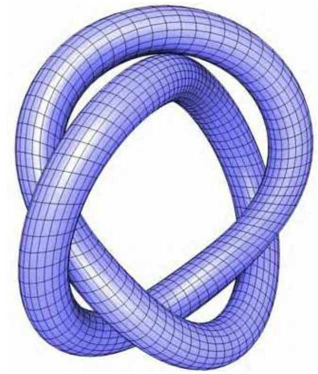
$$\frac{\mathrm{d}}{\mathrm{d}t} |\Omega(t)| = \int_{\Sigma(t)} V \,\mathrm{d}\mathcal{H}^{d-1} = 0,$$

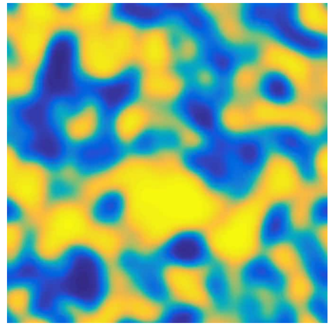
and hence

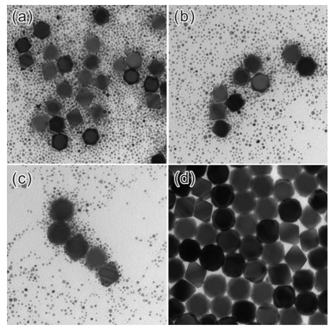
$$|\Omega(t)| = |\Omega(0)| =: m.$$

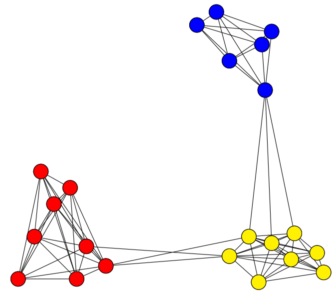
Motivation & Applications

- **Geometry:** Simplest area-decreasing geometric flow that preserves the enclosed volume [Gage '86, Huisken '87, Escher–Simonett '98]
- **PDE:** Appears as sharp-interface limit of reaction-diffusion systems [Rubinstein–Sternberg '92, Chen–Hilhorst–Logak '10, Bronsard–Stoth '97, L.–Simon '18, Takasao '17, '22]
- **Materials Science:** Basic model for Ostwald ripening; efficient method: thresholding/MBO scheme proposed by [Ruuth–Wetton '03], convergence proof by [L.–Swartz '17]
- **Data Science:** Arises as scaling limit of algorithms in unsupervised and semi-supervised learning [L.–Lelmi '21, L.–Lelmi '23], [Jacobs–Merkurjev–Esedoğlu '20, Krämer–L. '24+]

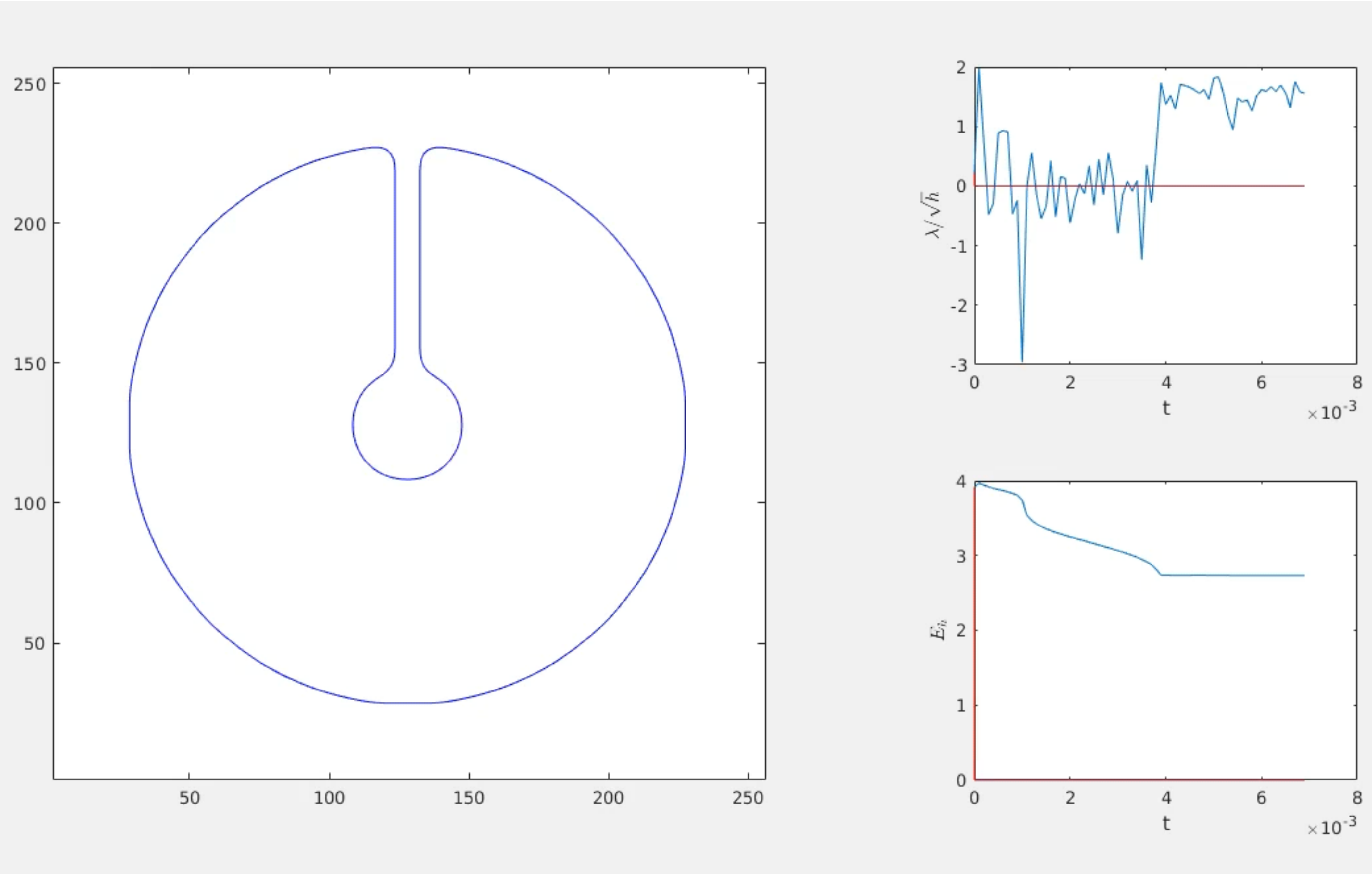








Simulation using thresholding



1st Main Result: Weak-strong uniqueness

Theorem (L. '22)

Classical solutions are unique and stable in the class of all distributional solutions.

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The result is non-trivial because:

- there are many examples of physical non-uniqueness when starting from singular configurations.
- There is no comparison principle, so uniqueness is subtle.
- Some weak solutions are fatally non-unique.

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Follows in two steps:

Theorem 1. *Any classical solution is a calibrated flow.*

Theorem 2. *Any calibrated flow is unique and stable in the class of all distributional solutions.*

Notion of solution

Classical solution: evolving surface $(\Sigma^*(t))_{t\in[0,T]}$ of class $C^{3,\alpha}$ satisfying

$$V^* = -H^* + \lambda^* \quad \text{everywhere on } \Sigma^*(t), \text{ for all } t \in [0,T].$$

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Distributional solution: $\chi = \chi(x,t) \in L^\infty((0,T); BV(\mathbb{R}^d; \{0,1\}))$, $V = V(x,t)$ $|\nabla\chi|$-measurable, $\lambda = \lambda(t)$ measurable s.t.

1. $\partial_t \chi = V|\nabla\chi|$
2. $(V-\lambda)\nabla\chi = -\nabla\cdot\left((I_d - \frac{\nabla\chi}{|\nabla\chi|}\otimes\frac{\nabla\chi}{|\nabla\chi|})|\nabla\chi|\right)$
3. For a.e. $T' \in (0,T)$

$$E[\chi(\cdot,T')] + \int_{\mathbb{R}^d\times(0,T')} V^2|\nabla\chi|\,\mathrm{d}t \le E[\chi(\cdot,0)].$$

4. For a.e. $t\in(0,T)$, $\int_{\mathbb{R}^d}\chi(\cdot,t)\,\mathrm{d}x = \int_{\mathbb{R}^d}\chi(\cdot,0)\,\mathrm{d}x$.
5. $\lambda \in L^2_{\mathrm{loc}}(0,T)$.

Calibrated flows

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Key players:

- ξ: extension of the normal vector field ν^*
- B: extension of the velocity vector field $V^*\nu^*$
- ϑ: (truncation of) the signed distance function $\mathrm{sdist}(\cdot, \Sigma^*(t))$.
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Conditions:

- B (almost) transports ξ and ϑ.
- $B \cdot \xi = -\nabla \cdot \xi + \lambda^* + O(\mathrm{dist}(\cdot, \Sigma^*(t)))$
- $|\xi| \leq \max\{1 - C\vartheta^2, 0\}$
- $\nabla \cdot B = O(\mathrm{dist}(\cdot, \Sigma^*(t)))$

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Ansatz: Take $B(\cdot,t) := \nabla\varphi$ where φ solves

$$\begin{cases} \Delta\varphi = 0 & \text{in } \Omega^*(t), \\ \nu^*(\cdot,t)\cdot\nabla\varphi = V^*(\cdot,t) & \text{on } \Sigma^*(t) = \partial\Omega^*(t). \end{cases}$$

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Idea: Monitor the relative energy

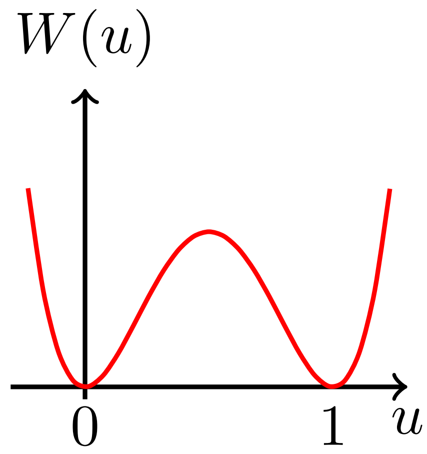
$$\mathcal{E}[\chi, \Sigma^*](t) := \int_{\mathbb{R}^d \times \{t\}} (1 - \xi \cdot \nu)|\nabla\chi|$$

and show $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} \leq C\mathcal{E}$.

Sharp-interface limit of Allen-Cahn

Phase-field [Golovaty '97] (based on [Rubinstein–Sternberg '92]) for preserved order-parameter:

$$\partial_t u_\varepsilon = \Delta u_\varepsilon - \frac{1}{\varepsilon^2} W'(u_\varepsilon) + \lambda_\varepsilon(t)\sqrt{2W(u_\varepsilon)},$$



where W is a double-well potential, e.g.,

$$W(u) = 3\sqrt{2}u^2(u-1)^2,$$

and

$$\lambda_\varepsilon(t) := -\frac{\int_{\mathbb{R}^d} \left(\Delta u_\varepsilon - \frac{1}{\varepsilon^2} W'(u_\varepsilon)\right)\sqrt{2W(u_\varepsilon)}\,\mathrm{d}x}{\int_{\mathbb{R}^d} 2W(u_\varepsilon)\,\mathrm{d}x}.$$

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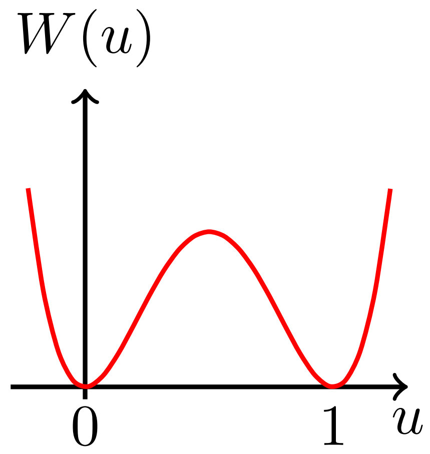

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Natural change of variables: $\phi(u) = \int_0^u \sqrt{2W(s)}\,\mathrm{d}s$.
Preserves the natural order-parameter $\psi_\varepsilon = \phi \circ u_\varepsilon$:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} \psi_\varepsilon(x,t)\ \mathrm{d}x = 0.$$

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In the limit $\varepsilon \downarrow 0$, we expect $u_\varepsilon(x,t) \to \chi_{\Omega(t)}(x)$, where $\Omega(t)$ is a solution to volume-preserving mean curvature flow.

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Theorem (Krömer & L. '23)

For well-prepared initial conditions, as long as a classical volume-preserving mean curvature flow $(\Sigma(t))_{t\in[0,T]}$ exists, we have the optimal convergence rate

$$\sup_{t\in[0,T]} \|\psi_\varepsilon(\cdot,t) - \chi_{\Omega(t)}\|_{L^1(\mathbb{R}^d)} \leq C(d,T,(\Omega(t))_{t\in[0,T]})\,\varepsilon.$$

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Also follows from a Gronwall argument, here for the phase-field relative entropy (cf. [Fischer–L.–Simon '20 SIMA])

$$\mathcal{E}_\varepsilon[u_\varepsilon,\Sigma](t) := E_\varepsilon[u_\varepsilon(\cdot,t)] - \int_{\mathbb{R}^d} \xi \cdot \nabla\psi_\varepsilon \,\mathrm{d}x.$$

Idea of proof: Calibration & relative entropy

Key challenge in closing the Gronwall estimate $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\varepsilon \leq C\mathcal{E}_\varepsilon$:
Estimating

$$\mathrm{Err}(t) = -\int_{\mathbb{R}^d\times\{t\}} \xi\cdot\nabla B\xi\big(\varepsilon|\nabla u_\varepsilon|^2 - |\nabla\psi_\varepsilon|\big)\,\mathrm{d}x.$$

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Idea: Have freedom in the choice of tangential component of B...
Ansatz: $B = (V\nu + X)\circ\pi_\Sigma$ where X is a tangential vector field on Σ satisfying

$$\mathrm{div}_\Sigma X = VH - \langle VH\rangle \quad \text{on } \Sigma.$$

This problem is underdetermined, so make the Ansatz $X = \nabla_\Sigma\varphi$ for a potential φ solving

$$\Delta_\Sigma\varphi = VH - \langle VH\rangle \quad \text{on } \Sigma.$$

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- Boundary contact can be taken into account in the unconstrained case [Hensel–L. '23 IUMJ] and is expected to work here, too.