Migrating elastic flows

Tatsuya Miura

Kyoto University

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Outline

Outline:

- Huisken's problem (unsolved): Can a closed planar elastic flow 'migrate' from the upper half-plane to the lower half-plane?
- Main result: Existence of migrating elastic flows for open planar curves, analytically and numerically.

References:

- Kemmochi–M., *Migrating elastic flows*, **J. Math. Pures Appl.** (2024)
- M.–Müller–Rupp, *Optimal thresholds for preserving embeddedness of elastic flows*, to appear in **Amer. J. Math.**
- M.–Yoshizawa, *General rigidity principles for stable and minimal elastic curves*, to appear in **J. Reine Angew. Math. (Crelle)**

Elastic flows

Elastic flows:

- Gradient flows of the bending energy, penalizing or prescribing length.
  - Polden (1996), Dziuk–Kuwert–Schätzle (2002), ...

Bending energy:

- A quantity that measures how much a curve is bending,

$$B[\gamma] := \int_\gamma k^2 ds,$$

  where γ is a planar curve and k is the (signed) curvature.
  - D. Bernoulli (1742), L. Euler (1744), ..., M. Born (1906), ...
- A critical point of B under fixed-length constraint, or equivalently of

$$E_\lambda := B + \lambda L \left(= \int (k^2 + \lambda) ds \right)$$

  for some $\lambda \in \mathbf{R}$, is called an elastica.

Elastic flows

Consider a one-parameter family of planar curves $\{\gamma(\cdot,t)\}_{t\geq 0}$.

Elastic flow (or length-penalized elastic flow):

- $L^2(ds)$-gradient flow of $E_\lambda = B + \lambda L$, where $\lambda > 0$ is a given constant.
- $\rightsquigarrow$ 4th order parabolic equation: in terms of normal velocity V,

$$V = -2k_{ss} - k^3 + \lambda k.$$

Length-preserving elastic flow:

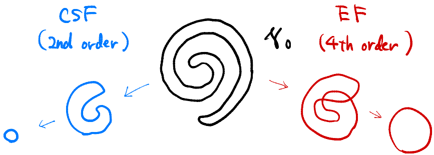
- $L^2(ds)$-gradient flow of B under the constraint $L[\gamma(\cdot,t)] = L[\gamma(\cdot,0)]$.
- $\rightsquigarrow$ the same equation, with the time-dependent nonlocal parameter

$$\lambda = \lambda[\gamma(\cdot,t)] = \frac{\int_{\gamma(\cdot,t)}(2k_{ss}k + k^4)ds}{\int_{\gamma(\cdot,t)} k^2 ds}.$$

Positivity breaking

Properties of elastic flows:

- Local well-posedness & global existence are well-known.
- 4th order $\rightsquigarrow$ lack of maximum principles $\rightsquigarrow$ positivity breaking.
- Examples of positivity; convexity, embeddedness, graphicality, ...



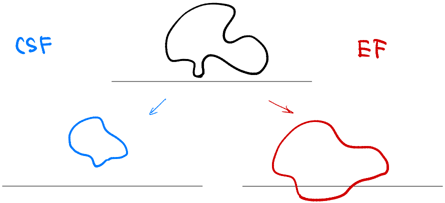
(See e.g. [M.–Müller–Rupp '24$^+$, Amer. J. Math.])

In this talk we focus on the "half-plane property" explained in the next slide.

Half-plane property and maximum principle

Half-plane property:

- Consider flows of closed curves, initially lying in a half-plane $H \subset \mathbf{R}^2$.
- Curve Shortening Flow ($V = k$) remains contained in H.
- Elastic Flow can protrude from H (lack of maximum principle).



Q. How much can an elastic flow get out of the half-plane H?

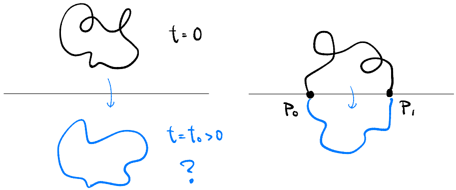
Huisken's problem

Huisken's problem:
Is there a closed planar length-penalized elastic flow γ such that

- $\gamma|_{t=0} \subset \{y \geq 0\}$ (contained in the upper half-plane at $t = 0$) but
- $\gamma|_{t=t_0} \subset \{y \leq 0\}$ (migrates to the lower half-plane) at some $t_0 > 0$?

Our results:

- Existence of migrating elastic flows of open curves, analytically in the length-preserving case, and numerically in both cases.



Main result

Elastic flows under the natural boundary condition:

- Let $\gamma : [0,1] \times [0,\infty) \to \mathbf{R}^2$ be an elastic flow of open curves $\gamma(\cdot, t)$.
- Natural BC: $\gamma(0,t) = (0,0)$, $\gamma(1,t) = (\ell, 0)$, and $k(0,t) = k(1,t) = 0$.

Theorem (Kemmochi–M. '24, J. Math. Pures Appl.)

There exists $c \in (0,1]$ such that for any $L > 0$ and $\ell \in (0, cL)$ there is a length-preserving elastic flow γ of length L subject to the NBC such that

- $\gamma((0,1) \times [0,t_0]) \subset \{y > 0\}$ *for some* $t_0 > 0$, *and*
- $\gamma((0,1) \times [t_1,\infty)) \subset \{y < 0\}$ *for some* $t_1 > t_0$.

Remark

The assumption taking c means that the endpoints are close to each other.

Numerical example 1: Length-preserving

- Numerical example corresponding to our theorem:

- Based on [Kemmochi–Miyatake–Sakakibara, arXiv:2208.00675].

Numerical example 2: Length-preserving

Conjecture:

- In the main theorem we can take $c = 1$ (the endpoints can be distant).

Numerical example 3: Length-preserving

Conjecture:

- Migration also occurs under (reflection) symmetry.

Numerical example 4: Length-penalized

Conjecture:

- Migration also occurs for the length-penalized flow (under NBC).

Numerical example 5: Length-penalized

Conjecture (soft statement):

- Large λ makes the flow "hard to migrate" (formally, $\lambda = \infty \Rightarrow$ CSF).

- This example does not migrate (seen after vertically stretched).

Numerical example 6: Length-penalized

Conjecture:

- Migration occurs both with length-penalization and symmetry.

(Small λ) (Large λ)

Summary and future directions

Summary:

- Existence of migrating elastic flows, in the case of length-preserving and endpoints close to each other.
- The proof is based on the variational structure of stationary solutions.
- More examples of migrating elastic flows numerically.

Open problems:

- Is there a length-penalized migrating elastic flow? (Numerically exists.)
- Is there a symmetric migrating elastic flow? (Numerically exists.)
- Is there an embedded migrating elastic flow? (No numerical results.)
- The case of closed curves is widely open (including Huisken's problem).

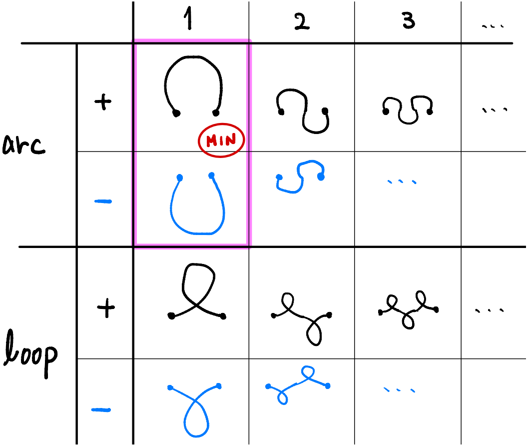
— Thank you.

Numerical example: closed curve

Sketch of proof (1/4)

Key tool is classification of stationary solutions of length L under the NBC.

- Global minimizers are convex arcs, unique up to reflection.
- All the other solutions are unstable. [M.–Yoshizawa '24$^+$, Crelle]

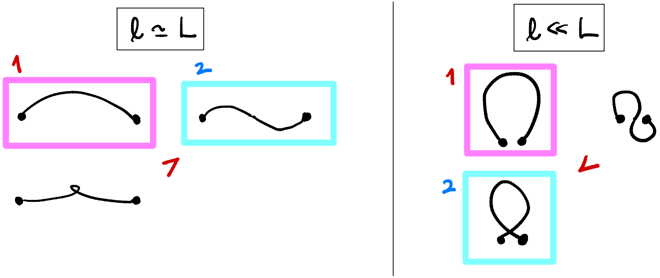


[Yoshizawa '22, DCDS] [M.–Yoshizawa '24$^+$, IUMJ]

Sketch of proof (2/4)

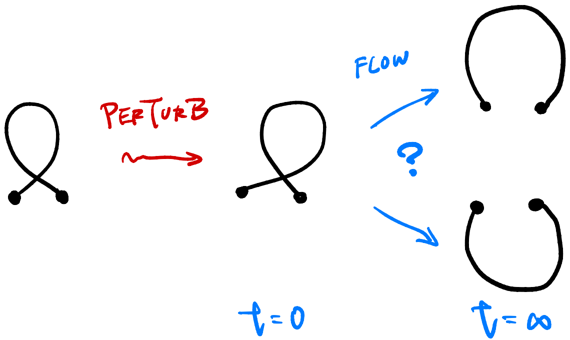
If the distance of the endpoints is small, that is, if $\ell \ll L$, then:

- The second smallest energy is attained by locally convex loops.



Sketch of proof (3/4)

- As the loop is unstable, we can find an energy decreasing perturbation. [M.–Yoshizawa '24 Crelle]
- As the flow decreases energy, the only candidates for the limit shape are global minimizers; namely, the upper arc and the lower arc.



Sketch of proof (4/4)

Consider the total curvature $TC[\gamma] = \int_\gamma k\, ds$. Again if $\ell \ll L$, then:

- The minimal bending energy B among all admissible curves γ with $TC[\gamma] = 0$ is strictly larger than the energy of the loop.
- That is, $\exists$ energy-barrier between the upper arc and the upper loop.