Error estimate for classical solutions to the heat equation in a moving thin domain and its limit equation

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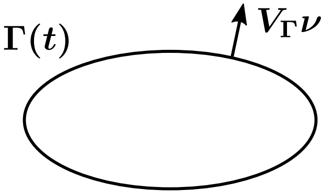
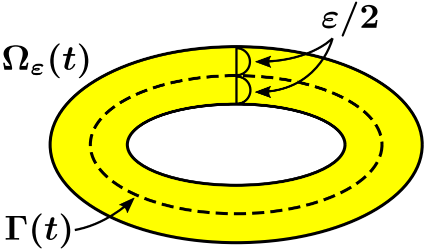
Moving Surface and Moving Thin Domain

- $\Gamma(t)$: given closed moving hypersurface in $\mathbb{R}^n$
  without changing topology ($n \geq 2$, $t \in [0, T]$)
  - $\nu(\cdot, t)$: unit outer normal vector field of $\Gamma(t)$
  - $V_\Gamma(\cdot, t)$: scalar outer normal velocity of $\Gamma(t)$
- $\Omega_\varepsilon(t)$: moving thin domain in $\mathbb{R}^n$ ($\varepsilon > 0$: small)

$$\Omega_\varepsilon(t) = \{y + r\nu(y, t) \mid y \in \Gamma(t),\ -\varepsilon/2 < r < \varepsilon/2\}$$





Heat Eq. in Moving Thin Domain

$$(H_\varepsilon)\begin{cases}\partial_t\rho^\varepsilon-\Delta\rho^\varepsilon=0 & \text{in}\quad Q_{\varepsilon,T}=\bigcup_{t\in(0,T]}\Omega_\varepsilon(t)\times\{t\}\\ \partial_{\nu_\varepsilon}\rho^\varepsilon+V_\varepsilon\rho^\varepsilon=0 & \text{on}\quad \partial_\ell Q_{\varepsilon,T}=\bigcup_{t\in(0,T]}\partial\Omega_\varepsilon(t)\times\{t\}\\ \rho^\varepsilon(\cdot,0)=\rho_0^\varepsilon & \text{in}\quad \Omega_\varepsilon(0)\end{cases}$$

- $\nu_\varepsilon(\cdot,t)$: unit outer normal vector field of $\partial\Omega_\varepsilon(t)$
- $V_\varepsilon(\cdot,t)$: scalar outer normal velocity of $\partial\Omega_\varepsilon(t)$
- Neumann type B.C. with additional term $V_\varepsilon\rho^\varepsilon$

$$\Leftrightarrow\quad \frac{d}{dt}\int_{\Omega_\varepsilon(t)}\rho^\varepsilon\,dx=0,\quad t\in(0,T)$$

- Our interest: What happens when the thickness $\varepsilon\to0$?

(Thin-film Limit Problem)

Previous Works on Moving Thin Domains

- Elliott–Stinner (2009): diffuse interface approximation in

  $$\Omega_\varepsilon(t) \xrightarrow{\varepsilon\to 0} \Gamma(t)\text{: moving hypersurface}$$

  of an advection-diffusion eq. on $\Gamma(t)$
- Elliott–Stinner–Styles–Welford (2011):
  numerical computation of the diffuse interface model
- Pereira–Silva (2013): reaction-diffusion eq. in

  $$\Omega_\varepsilon(t) = \{(x', x_n) \in \mathbb{R}^n \mid x' \in \omega,\ 0 < x_n < \varepsilon g(x,t)\}$$
  $$\xrightarrow{\varepsilon\to 0} \omega \subset \mathbb{R}^{n-1}\text{: stationary domain}$$

- M. (2017): rigorous derivation of a limit eq. of (H_ε) in L^2

Thin-film Limit of (H_ε) in the L^2-framework

$$(H_\varepsilon)\ \partial_t\rho^\varepsilon - \Delta\rho^\varepsilon = 0 \ \text{in}\ Q_{\varepsilon,T},\ \partial_{\nu_\varepsilon}\rho^\varepsilon + V_\varepsilon\rho^\varepsilon = 0 \ \text{on}\ \partial_\ell Q_{\varepsilon,T}$$

M. (2017, Interfaces Free Bound.)

- For an L^2-weak sol. ρ^ε to (H_ε), when $\varepsilon \to 0$,

  Average of ρ^ε in thin direction $\to \exists\eta$ weakly on $\Gamma(t)$

- η is a unique L^2-weak sol. to the limit eq.

$$(H_0)\begin{cases}\partial^\circ\eta - V_\Gamma H\eta - \Delta_\Gamma\eta = 0 \ \text{on}\ S_T = \bigcup_{t\in(0,T]}\Gamma(t)\times\{t\} \\ \eta(\cdot,0) = \eta_0 \ \text{on}\ \Gamma(0)\end{cases}$$

  - $\partial^\circ = \partial_t + V_\Gamma\nu\cdot\nabla$: normal time derivative
  - H: mean curvature, Δ_Γ: Laplace–Beltrami op.

- (H_0) is a diffusion eq. on $\Gamma(t)$ (cf. Dziuk–Elliott (2007))

M. (2017, Interfaces Free Bound.)

- L^2-difference estimate for ρ^ε and η

$$\varepsilon^{-1/2}\|\rho^\varepsilon - \bar{\eta}\|_{L^2(Q_{\varepsilon,T})} \leq c\Big(\varepsilon^{-1/2}\|\rho_0^\varepsilon - \bar{\eta}_0\|_{L^2(\Omega_\varepsilon(0))} + \varepsilon\|\eta_0\|_{L^2(\Gamma(0))}\Big)$$

  - ▷ $\bar{\eta}(y + r\nu(y,t), t) = \eta(y,t)$ $(y \in \Gamma(t),\ |r| < \varepsilon/2)$

- Estimate in $L^2(Q_{\varepsilon,T})$ has ambiguity due to $|\Omega_\varepsilon(t)| = O(\varepsilon)$
  $\rightarrow$ To avoid it, we take the scaled norm $\varepsilon^{-1/2}\|\cdot\|_{L^2(Q_{\varepsilon,T})}$

Aim of This Talk

Another approach to avoid ambiguity due to $|\Omega_\varepsilon(t)| = O(\varepsilon)$:

- Difference estimate in the sup-norm for classical solutions

Main Result

Theorem 1 (M., Interfaces Free Bound., 2023)

Let ρ^ε and η be classical sols. to (H_ε) and (H_0). Then,

$$\|\rho^\varepsilon - \bar{\eta}\|_{\mathcal{B}(\overline{Q_{\varepsilon,T}})} \leq c\Big(\|\rho_0^\varepsilon - \bar{\eta}_0\|_{\mathcal{B}(\overline{\Omega_\varepsilon(0)})} + \varepsilon\|\eta\|_{\mathcal{B}^{2,1}(\overline{S_T})}\Big)$$

- $\|\psi\|_{\mathcal{B}(\Omega)} = \sup_\Omega |\psi|$ for a bounded ψ on $\Omega \subset \mathbb{R}^k$ $(k \geq 1)$
- $\|\eta\|_{\mathcal{B}^{2,1}(\overline{S_T})} = \|\eta\|_{\mathcal{B}(\overline{S_T})} + \|\partial^\circ\eta\|_{\mathcal{B}(\overline{S_T})} + \sum_{k=1,2}\|\nabla_\Gamma^k\eta\|_{\mathcal{B}(\overline{S_T})}$ $\Big(\|\eta\|_{\mathcal{B}^{2,1}(\overline{S_T})} \leq c\|\eta_0\|_{C^{2+\alpha}(\Gamma(0))},\ 0<\alpha<1\Big)$

$$(H_\varepsilon)\begin{cases}\partial_t\rho^\varepsilon - \Delta\rho^\varepsilon = 0 \text{ in } Q_{\varepsilon,T},\ \partial_{\nu_\varepsilon}\rho^\varepsilon + V_\varepsilon\rho^\varepsilon = 0 \text{ on } \partial_\ell Q_{\varepsilon,T} \\ \rho^\varepsilon(\cdot,0) = \rho_0^\varepsilon \text{ in } \Omega_\varepsilon(0)\ \ \big(\rho_0^\varepsilon \in C(\overline{\Omega_\varepsilon(0)})\big)\end{cases}$$

$$(H_0)\begin{cases}\partial^\circ\eta - V_\Gamma H\eta - \Delta_\Gamma\eta = 0 \text{ on } S_T \\ \eta(\cdot,0) = \eta_0 \text{ on } \Gamma(0)\ \ \big(\eta_0 \in C(\Gamma(0))\big)\end{cases}$$

Idea of Proof

(H_ε) $\partial_t\rho^\varepsilon - \Delta\rho^\varepsilon = 0$ in $Q_{\varepsilon,T}$, $\partial_{\nu_\varepsilon}\rho^\varepsilon + V_\varepsilon\rho^\varepsilon = 0$ on $\partial_\ell Q_{\varepsilon,T}$

(H_0) $\partial^\circ\eta - V_\Gamma H\eta - \Delta_\Gamma\eta = 0$ on S_T

Idea is standard, but we need to be always careful about ε:

(1) Construct an approximate solution ρ_η^ε ($\approx \bar{\eta}$) to (H_ε)

(2) Estimate $\rho^\varepsilon - \rho_\eta^\varepsilon$ by the maximum principle for (H_ε)

In the actual proof, we first derive an a priori estimate for

$$(\widetilde{H_\varepsilon}) \begin{cases} \partial_t\tilde{\rho}^\varepsilon - \Delta\tilde{\rho}^\varepsilon = f_\eta^\varepsilon & \text{in } Q_{\varepsilon,T} \\ \partial_{\nu_\varepsilon}\tilde{\rho}^\varepsilon + V_\varepsilon\tilde{\rho}^\varepsilon = \psi_\eta^\varepsilon & \text{on } \partial_\ell Q_{\varepsilon,T} \\ \tilde{\rho}^\varepsilon(\cdot,0) = \tilde{\rho}_0^\varepsilon & \text{in } \Omega_\varepsilon(0) \end{cases} \quad \begin{pmatrix} \tilde{\rho}^\varepsilon = \rho^\varepsilon - \rho_\eta^\varepsilon \\ f_\eta^\varepsilon, \psi_\eta^\varepsilon\text{: errors} \end{pmatrix}$$

in order to determine what is a suitable approx. sol. ρ_η^ε.

A Priori Estimate with Explicit Dependence on ε

Theorem 2 (M., Interfaces Free Bound., 2023)

If $\tilde{\rho}^\varepsilon$ is a classical sol. to $(\widetilde{H}_\varepsilon)$, then $\exists c > 0$ indep. of ε s.t.

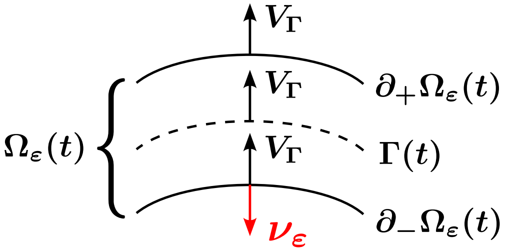
$$\|\tilde{\rho}^\varepsilon\|_{\mathcal{B}(\overline{Q_{\varepsilon,T}})} \leq c\Big(\|\tilde{\rho}_0^\varepsilon\|_{\mathcal{B}(\overline{\Omega_\varepsilon(0)})} + \|f_\eta^\varepsilon\|_{\mathcal{B}(Q_{\varepsilon,T})} + \varepsilon^{-1}\|\psi_\eta^\varepsilon\|_{\mathcal{B}(\partial_\ell Q_{\varepsilon,T})}\Big)$$

Difficulty: we argue by the maximum principle, but

$$V_\varepsilon = \pm V_\Gamma \quad \text{on} \quad \partial_\pm\Omega_\varepsilon(t)$$

in the boundary condition

$$\partial_{\nu_\varepsilon}\tilde{\rho}^\varepsilon + V_\varepsilon\tilde{\rho}^\varepsilon = \psi_\eta^\varepsilon \quad \text{on} \quad \partial_\pm\Omega_\varepsilon(t)$$



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Idea: for $x = y + r\nu(y,t) \in \overline{\Omega_\varepsilon(t)}$ ($y \in \Gamma(t)$, $|r| \leq \varepsilon/2$), let

$$\Phi_\varepsilon(x,t) = rV_\Gamma(y,t) - (r+\varepsilon/2)(r-\varepsilon/2)$$

Then, $\zeta^\varepsilon(x,t) = e^{\Phi_\varepsilon(x,t)}\tilde{\rho}^\varepsilon(x,t)$ satisfies

$$\begin{cases} \partial_t\zeta^\varepsilon - \Delta\zeta^\varepsilon + \mathrm{b}^\varepsilon\cdot\nabla\zeta^\varepsilon + C^\varepsilon\zeta^\varepsilon = e^{\Phi_\varepsilon}f^\varepsilon \text{ in } Q_{\varepsilon,T} \\ \partial_{\nu_\varepsilon}\zeta^\varepsilon + \varepsilon\zeta^\varepsilon = e^{\Phi_\varepsilon}\psi^\varepsilon \text{ on } \partial_\ell Q_{\varepsilon,T} \\ \zeta^\varepsilon(\cdot,0) = e^{\Phi_\varepsilon(\cdot,0)}\tilde{\rho}_0^\varepsilon \text{ in } \Omega_\varepsilon(0) \end{cases}$$

with $|C^\varepsilon| \leq C_1$ (const. indep. of ε) $\Rightarrow$ Standard argument

Choice of Suitable Approximate Solution

- By Theorem 2, the approx. sol. ρ_η^ε ($\approx \bar{\eta}$) should satisfy

$$\begin{cases} \partial_t \rho_\eta^\varepsilon - \Delta \rho_\eta^\varepsilon = O(\varepsilon) & \text{in} \quad Q_{\varepsilon,T} \\ \partial_{\nu_\varepsilon} \rho_\eta^\varepsilon + V_\varepsilon \rho_\eta^\varepsilon = O(\varepsilon^2) & \text{on} \quad \partial_\ell Q_{\varepsilon,T} \end{cases}$$

However, $\bar{\eta}$ itself does not satisfy this due to $V_\varepsilon \rho_\eta^\varepsilon$ in B.C.!

- To find ρ_η^ε, we substitute the asymptotic expansion

$$\rho^\varepsilon(y + \varepsilon z \nu(y,t), t) = \eta_0(y,t,z) + \varepsilon \eta_1(y,t,z) + \cdots$$
$$(y,t) \in S_T, \ z \in [-1/2, 1/2]$$

for (H_ε) and determine η_0, η_1, and η_2:

$$\begin{cases} O(\varepsilon^{-2}) \text{ in } Q_{\varepsilon,T}, \ O(\varepsilon^{-1}) \text{ on } \partial_\ell Q_{\varepsilon,T} \Rightarrow \eta_0(z) = \eta_0 \\ O(\varepsilon^{-1}) \text{ in } Q_{\varepsilon,T}, \ O(1) \text{ on } \partial_\ell Q_{\varepsilon,T} \Rightarrow \eta_1(z) = -zV_\Gamma \eta_0 \end{cases}$$

- $O(1)$ in $Q_{\varepsilon,T}$, $O(\varepsilon)$ on $\partial_\ell Q_{\varepsilon,T} \Rightarrow$ Equation for $\eta_2(z)$:

$$\begin{cases} \partial_z^2 \eta_2(z) = \partial^\circ \eta_0 - \Delta_\Gamma \eta_0 - V_\Gamma H \eta_0 + V_\Gamma^2 \eta_0 \ \ (|z| < 1/2) \\ \partial_z \eta_2(\pm 1/2) = \pm V_\Gamma^2 \eta_0 / 2 \end{cases}$$

- Since $\int_{-1/2}^{1/2} \partial_z^2 \eta_2(z)\, dz = [\partial_z \eta_2(z)]_{-1/2}^{1/2}$, we must have

$$\partial^\circ \eta_0 - V_\Gamma H \eta_0 - \Delta_\Gamma \eta_0 = 0 \ \text{ on } \ S_T \ \ (\text{Limit eq.})$$

  In this case, $\eta_2(z) = z^2 V_\Gamma^2 \eta_0 / 2$.

- Therefore, the suitable approx. sol. ρ_η^ε to (H_ε) is

$$\begin{aligned} \rho_\eta^\varepsilon(y + \varepsilon z \nu(y,t), t) &= \textstyle\sum_{k=0}^2 \varepsilon^k \eta_k(y,t,z) \\ &= \eta_0(y,t) - \varepsilon z (V_\Gamma \eta_0)(y,t) + \{(\varepsilon z)^2/2\}(V_\Gamma^2 \eta_0)(y,t) \\ &\qquad (y,t) \in \overline{S_T},\ z \in [-1/2, 1/2] \end{aligned}$$

Thank you for your attention!

Definitions of Classical Solutions

- ρ^ε is a classical sol. to $(H_\varepsilon) \overset{\text{def}}{\Leftrightarrow}$

$$\rho^\varepsilon \in C(\overline{Q_{\varepsilon,T}}), \quad \partial_i \rho^\varepsilon \in C(Q_{\varepsilon,T} \cup \partial_\ell Q_{\varepsilon,T}),$$
$$\partial_t \rho^\varepsilon, \partial_i \partial_j \rho^\varepsilon \in C(Q_{\varepsilon,T})$$

  and ρ^ε satisfies (H_ε) at each point of $\overline{Q_{\varepsilon,T}}$

- η is a classical sol. to $(H_0) \overset{\text{def}}{\Leftrightarrow}$

$$\eta \in C(\overline{S_T}), \quad \partial^\circ \eta, \underline{D}_i \eta, \underline{D}_i \underline{D}_j \eta \in C(S_T)$$

  and η satisfies (H_0) at each point of $\overline{S_T}$

A Priori Estimate with Explicit Dependence on ε

Theorem 2 (M., Interfaces Free Bound., 2023)

If $\tilde{\rho}^\varepsilon$ is a classical sol. to $(\widetilde{H_\varepsilon})$, then $\exists c > 0$ indep. of ε s.t.

$$\|\tilde{\rho}^\varepsilon\|_{\mathcal{B}(\overline{Q_{\varepsilon,T}})} \leq c\Big(\|\tilde{\rho}_0^\varepsilon\|_{\mathcal{B}(\overline{\Omega_\varepsilon(0)})} + \|f_\eta^\varepsilon\|_{\mathcal{B}(Q_{\varepsilon,T})} + \varepsilon^{-1}\|\psi_\eta^\varepsilon\|_{\mathcal{B}(\partial_\ell Q_{\varepsilon,T})}\Big)$$

<u>Idea of Proof</u>

- For $x = y + r\nu(y,t) \in \overline{\Omega_\varepsilon(t)}$ with $y \in \Gamma(t)$, $|r| \leq \varepsilon/2$, let

$$\Phi_\varepsilon(x,t) = rV_\Gamma(y,t) - (r+\varepsilon/2)(r-\varepsilon/2)$$

  Then, $\zeta^\varepsilon(x,t) = e^{\Phi_\varepsilon(x,t)}\tilde{\rho}^\varepsilon(x,t)$ satisfies

$$\begin{cases} \partial_t\zeta^\varepsilon - \Delta\zeta^\varepsilon + \mathrm{b}^\varepsilon \cdot \nabla\zeta^\varepsilon + C^\varepsilon\zeta^\varepsilon = e^{\Phi_\varepsilon} f^\varepsilon & \text{in} \quad Q_{\varepsilon,T} \\ \partial_{\nu_\varepsilon}\zeta^\varepsilon + \varepsilon\zeta^\varepsilon = e^{\Phi_\varepsilon}\psi^\varepsilon & \text{on} \quad \partial_\ell Q_{\varepsilon,T} \\ \zeta^\varepsilon(\cdot,0) = e^{\Phi_\varepsilon(\cdot,0)}\tilde{\rho}_0^\varepsilon & \text{in} \quad \Omega_\varepsilon(0) \end{cases}$$

$$\begin{cases} \partial_t \zeta^\varepsilon - \Delta\zeta^\varepsilon + \mathrm{b}^\varepsilon \cdot \nabla\zeta^\varepsilon + C^\varepsilon\zeta^\varepsilon = e^{\Phi_\varepsilon} f^\varepsilon & \text{in} \quad Q_{\varepsilon,T} \\ \partial_{\nu_\varepsilon}\zeta^\varepsilon + \varepsilon\zeta^\varepsilon = e^{\Phi_\varepsilon}\psi^\varepsilon & \text{on} \quad \partial_\ell Q_{\varepsilon,T} \\ \zeta^\varepsilon(\cdot,0) = e^{\Phi_\varepsilon(\cdot,0)}\tilde{\rho}_0^\varepsilon & \text{in} \quad \Omega_\varepsilon(0) \end{cases}$$

- Moreover, $\exists C_1 > 0$ indep. of ε s.t. $|C^\varepsilon| \leq C_1$ in $Q_{\varepsilon,T}$.
- Hence, $Z_\pm^\varepsilon(x,t) = \pm e^{-(C_1+1)t}\zeta^\varepsilon(x,t) - M^\varepsilon$ with

$$M^\varepsilon = \|e^{\Phi_\varepsilon(\cdot,0)}\tilde{\rho}_0^\varepsilon\|_{\mathcal{B}(\overline{\Omega_\varepsilon(0)})} + \|e^{\Phi_\varepsilon} f^\varepsilon\|_{\mathcal{B}(Q_{\varepsilon,T})} + \varepsilon^{-1}\|e^{\Phi_\varepsilon}\psi^\varepsilon\|_{\mathcal{B}(\partial_\ell Q_{\varepsilon,T})}$$

satisfies ($D^\varepsilon(x,t) = C^\varepsilon(x,t) + C_1 + 1 \geq 1$)

$$\begin{cases} \partial_t Z_\pm^\varepsilon - \Delta Z_\pm^\varepsilon + \mathrm{b}^\varepsilon \cdot \nabla Z_\pm^\varepsilon + D^\varepsilon Z_\pm^\varepsilon \leq 0 \ \text{ in } \ Q_{\varepsilon,T} \\ \partial_{\nu_\varepsilon} Z_\pm^\varepsilon + \varepsilon Z_\pm^\varepsilon \leq 0 \ \text{ on } \ \partial_\ell Q_{\varepsilon,T}, \quad Z_\pm^\varepsilon(\cdot,0) \leq 0 \ \text{ in } \ \Omega_\varepsilon(0) \end{cases}$$

$\Rightarrow Z_\pm^\varepsilon \leq 0 \Rightarrow$ Estimate for $\zeta^\varepsilon = e^{\Phi_\varepsilon}\tilde{\rho}^\varepsilon \Rightarrow$ Theorem 2

A Priori Estimate is Optimal

- Assume that $\exists c > 0$ indep. of ε s.t.

$$(\flat)\ \|\tilde{\rho}^\varepsilon\|_{\mathcal{B}(\overline{Q_{\varepsilon,T}})} \leq c\Big(\|\tilde{\rho}_0^\varepsilon\|_{\mathcal{B}(\overline{\Omega_\varepsilon(0)})} + \|f^\varepsilon\|_{\mathcal{B}(Q_{\varepsilon,T})} + \|\psi^\varepsilon\|_{\mathcal{B}(\partial_\ell Q_{\varepsilon,T})}\Big)$$

  for all classical sol. $\tilde{\rho}^\varepsilon$ to

$$\begin{cases} \partial_t\tilde{\rho}^\varepsilon - \Delta\tilde{\rho}^\varepsilon = f^\varepsilon & \text{in} \quad Q_{\varepsilon,T} \\ \partial_{\nu_\varepsilon}\tilde{\rho}^\varepsilon + V_\varepsilon\tilde{\rho}^\varepsilon = \psi^\varepsilon & \text{on} \quad \partial_\ell Q_{\varepsilon,T} \\ \tilde{\rho}^\varepsilon(\cdot,0) = \tilde{\rho}_0^\varepsilon & \text{in} \quad \Omega_\varepsilon(0) \end{cases}$$

- For any smooth $\zeta, \zeta_2 \colon \overline{S_T} \to \mathbb{R}$, let

$$\tilde{\rho}^\varepsilon(x,t) = \zeta(y,t) - r(V_\Gamma\zeta)(y,t) + (r^2/2)\zeta_2(y,t)$$
$$x = y + r\nu(y,t) \in \overline{\Omega_\varepsilon(t)},\ (y,t) \in \overline{S_T},\ |r| \leq \varepsilon/2$$

$$\tilde{\rho}^\varepsilon(x,t) = \zeta(y,t) - r(V_\Gamma\zeta)(y,t) + (r^2/2)\zeta_2(y,t)$$
$$x = y + r\nu(y,t) \in \overline{\Omega_\varepsilon(t)},\ (y,t) \in \overline{S_T},\ |r| \le \varepsilon/2$$

- Then, $\psi^\varepsilon = \partial_{\nu_\varepsilon}\tilde{\rho}^\varepsilon + V_\varepsilon\tilde{\rho}^\varepsilon$ on $\partial_\ell Q_{\varepsilon,T}$ satisfies

$$\begin{aligned}\psi^\varepsilon(x,t) &= \pm(\varepsilon/2)(\zeta_2 - V_\Gamma\zeta)(y,t) + (\varepsilon^2/8)(V_\Gamma\zeta_2)(y,t) \\ &= O\Big(\varepsilon\Big(\|\zeta\|_{\mathcal{B}(\overline{S_T})} + \|\zeta_2\|_{\mathcal{B}(\overline{S_T})}\Big)\Big)\end{aligned}$$

- Also, $f^\varepsilon = \partial_t\tilde{\rho}^\varepsilon - \Delta\tilde{\rho}^\varepsilon$ in $Q_{\varepsilon,T}$ satisfies

$$\begin{aligned}f^\varepsilon(x,t) &= (\partial^\circ\zeta + V_\Gamma^2 H\zeta - V_\Gamma H\zeta - \Delta_\Gamma\zeta - \zeta_2)(y,t) \\ &\quad + O\Big(\varepsilon\Big(\|\zeta\|_{\mathcal{B}^{2,1}(\overline{S_T})} + \|\zeta_2\|_{\mathcal{B}^{2,1}(\overline{S_T})}\Big)\Big)\end{aligned}$$

- Thus, if $\zeta_2 = \partial^\circ\zeta + V_\Gamma^2 H\zeta - V_\Gamma H\zeta - \Delta_\Gamma\zeta$ on $\overline{S_T}$, then

$$f^\varepsilon(x,t) = O\Big(\varepsilon\Big(\|\zeta\|_{\mathcal{B}^{2,1}(\overline{S_T})} + \|\zeta_2\|_{\mathcal{B}^{2,1}(\overline{S_T})}\Big)\Big)$$

$$\tilde{\rho}^\varepsilon(x,t) = \zeta(y,t) - r(V_\Gamma\zeta)(y,t) + (r^2/2)\zeta_2(y,t)$$

- Now we apply (♭) to $\tilde{\rho}^\varepsilon$ to get

$$\begin{aligned}\|\tilde{\rho}^\varepsilon\|_{\mathcal{B}(\overline{Q_{\varepsilon,T}})} &\leq c\Big(\|\tilde{\rho}_0^\varepsilon\|_{\mathcal{B}(\overline{\Omega_\varepsilon(0)})} + \|f^\varepsilon\|_{\mathcal{B}(Q_{\varepsilon,T})} + \|\psi^\varepsilon\|_{\mathcal{B}(\partial_\ell Q_{\varepsilon,T})}\Big) \\ &\leq c\left\{\|\tilde{\rho}_0^\varepsilon\|_{\mathcal{B}(\overline{\Omega_\varepsilon(0)})} + \varepsilon\left(\|\zeta\|_{\mathcal{B}^{2,1}(\overline{S_T})} + \|\zeta_2\|_{\mathcal{B}^{2,1}(\overline{S_T})}\right)\right\}\end{aligned}$$

From this, we further deduce that

$$\|\zeta\|_{\mathcal{B}(\overline{S_T})} \leq c\left\{\|\zeta(\cdot,0)\|_{\mathcal{B}(\Gamma(0))} + \varepsilon\left(\|\zeta\|_{\mathcal{B}^{2,1}(\overline{S_T})} + \|\zeta_2\|_{\mathcal{B}^{2,1}(\overline{S_T})}\right)\right\}$$

- Letting $\varepsilon \to 0$, we obtain

$$\|\zeta\|_{\mathcal{B}(\overline{S_T})} \leq c\|\zeta(\cdot,0)\|_{\mathcal{B}(\Gamma(0))}$$

for any smooth $\zeta\colon \overline{S_T} \to \mathbb{R}$ (obviously absurd!)

Zeroth Order Terms of Time Derivative and Laplacian

- $d(\cdot,t)$: signed distance from $\Gamma(t)$
- $H(\cdot,t)$: mean curvature of $\Gamma(t)$
- For $\eta\colon S_T \to \mathbb{R}$ and $y \in \Gamma(t)$, $|r| < \varepsilon/2$,

$$\bar{\eta}(x,t) = \eta(y,t), \quad x = y + r\nu(y,t) \in \Omega_\varepsilon(t)$$

- Then, for $(d^k\bar{\eta})(x,t) = d(x,t)^k\bar{\eta}(x,t)$ with $k = 0,1,2$,

$$\left.\begin{array}{r} \left\|\partial_t\bar{\eta} - \overline{\partial^\circ\eta}\right\|_{\mathcal{B}(Q_{\varepsilon,T})} \\ \left\|\Delta\bar{\eta} - \overline{\Delta_\Gamma\eta}\right\|_{\mathcal{B}(Q_{\varepsilon,T})} \\ \left\|\Delta(d\bar{\eta}) - \left(-\overline{H\eta}\right)\right\|_{\mathcal{B}(Q_{\varepsilon,T})} \\ \left\|\Delta(d^2\bar{\eta}) - 2\bar{\eta}\right\|_{\mathcal{B}(Q_{\varepsilon,T})} \end{array}\right\} \leq c\varepsilon\|\eta\|_{\mathcal{B}^{2,1}(S_T)}$$