Gradient continuity for very singular equations with one-Laplacian

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$(1,\, p)$-Laplace regularity

Let $u = u(x)$ or $u = u(x,\, t)$ (real-valued) satisfy

$\begin{cases} x = (x_1, \cdots, x_n) \\ n \geq 2 \end{cases}$

$p \in (1, \infty)$

$$\begin{aligned} \Delta_1 u + \Delta_p u &= 0, \quad \text{(SE)} \\ \partial_t u - (\Delta_1 u + \Delta_p u) &= 0, \quad \text{(SP)} \end{aligned}$$

in a weak sense.

$\Delta_s u := \mathrm{div}\,(|\nabla u|^{s-2} \nabla u)$

for $s \in [1, \infty)$

- When $s = 1$, use $Z \in \partial |\cdot| (\nabla u)$ a.e. instead of $\dfrac{\nabla u}{|\nabla u|}$.

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Main Result: $\nabla u = (\partial_{x_j} u)_{j=1,\,\dots\,,\,n} \in C^0$.

(SE): T.; Adv. Calc. Var. (eq.) & Math. Ann. (system)

(SP): T.; arXiv:2306.06868 & 2402.04951 (submitted)

Note : $\nabla u \in C^\alpha$ or $\nabla u \in C^{\alpha, \alpha/2}$ still remains open ...

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p=2 , Bingham flow p=3 , crystal surface

Sources of $\Delta_1 + \Delta_p$: Duvaut–Lions ('76), Spohn ('93)

Strategy

As $\nabla u \to 0$, $(\Delta_1 + \Delta_p)u$ loses its uniform ellipticity.

Note: (Ellipticity Ratio) $\simeq 1 + \underbrace{|\nabla u|^{1-p}}_{\text{effect of 1-Laplacian}} \longrightarrow \infty$ as $\nabla u \to 0$

$$\frac{\text{max e.v. of } \nabla^2 E(\nabla u)}{\text{min e.v. of } \nabla^2 E(\nabla u)} \quad \text{with } E(z) = |z| + \frac{|z|^p}{p}$$

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Note: (Ellipticity Ratio) $\simeq 1 + |\nabla u|^{1-p}$ $\leq 1 + \delta^{1-p} < \infty$

on $\{ |\nabla u| \geq \delta \}$

$\|$

$\operatorname{spt} \mathcal{G}_\delta(\nabla u)$

Claim: $\forall \delta \in (0,\, 1)$, the truncated gradient

$$\mathcal{G}_\delta(\nabla u) := (|\nabla u| - \delta)_+ \frac{\nabla u}{|\nabla u|}$$

is locally Hölder continuous, depending on δ.

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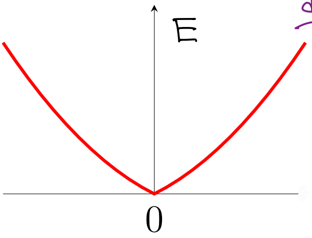
Then, $\mathcal{G}_\delta(\nabla u) \overset{\text{unif.}}{\to} \nabla u \in C^0$.

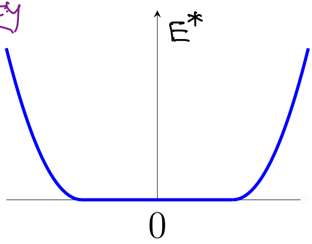
Singular $\leftrightarrow$ Degenerate

Operator : $\mathrm{div}(\nabla E(\nabla u))$ $\qquad$ $\mathrm{div}(\nabla E^*(\nabla v))$

duality

Energy density :





Regularity :

$$\mathcal{G}_0(\nabla u) = \nabla u \in C^0 \qquad \mathcal{G}_1(\nabla v) \in C^0$$

Gradient continuity for degenerate problems is shown by

- Santambrogio–Vespri ('10), Colombo–Figalli ('14) $\to$ elliptic eq.,
- Bögelein–Duzaar–Giova–Passarelli di Napoli ('22$\sim$)
  $\to$ elliptic & parabolic system.