Ideal curve flow with constraints on length

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Bending energy

Critical points

$$\frac{1}{2}\int_\gamma k^2\,ds$$

Gradient flows

Stability

k : curvature, s : arc length parameter.

"Generalized" bending energy

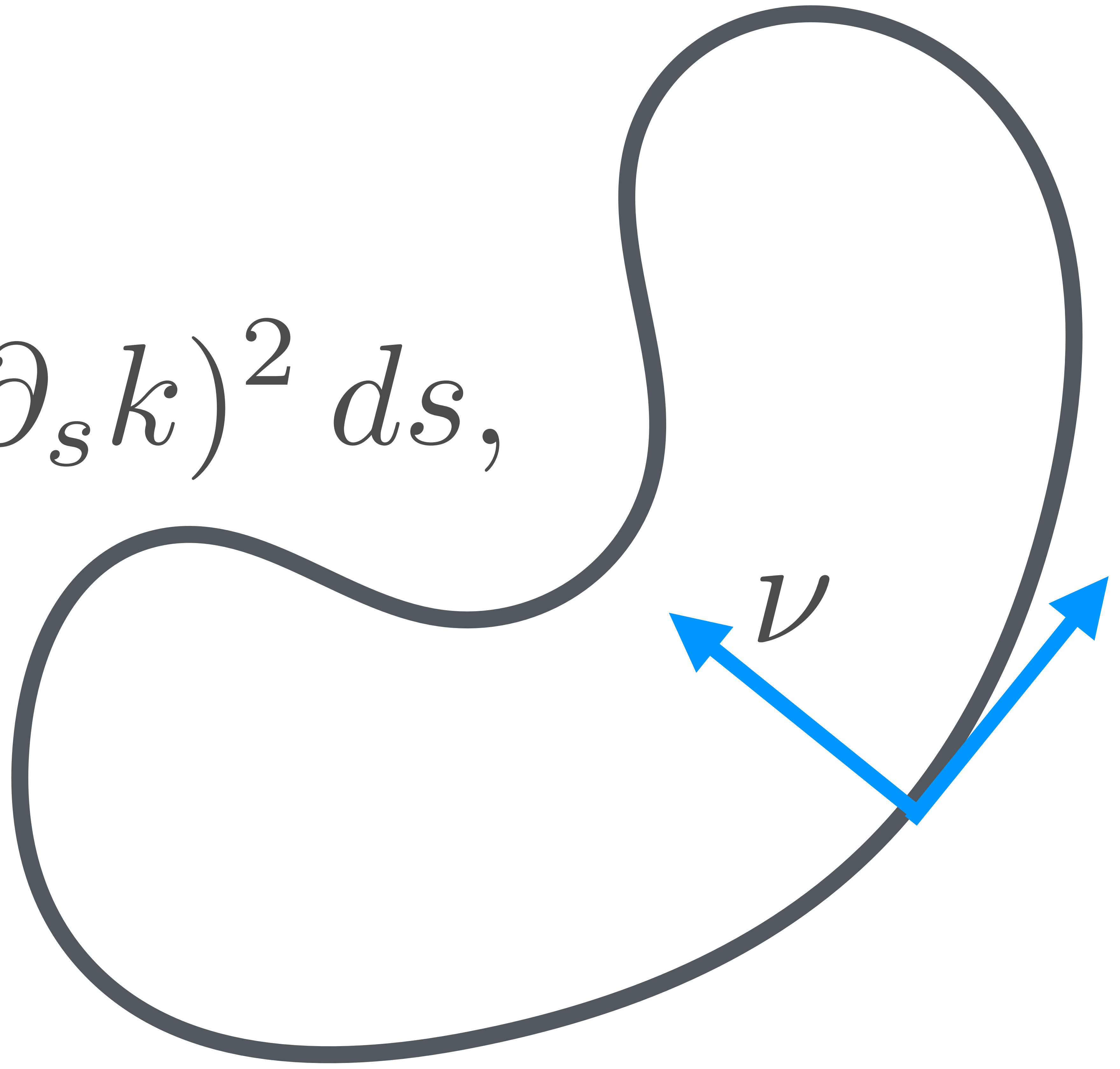
$$\frac{1}{p}\int_\gamma |k|^p\,ds$$

$(p > 1)$

$$\frac{1}{p}\int_\gamma |\partial_s k|^p\,ds$$

Ideal functional

$$E(\gamma) := \frac{1}{2}\int_\gamma (\partial_s k)^2\, ds,$$



for $\gamma : S^1 := \mathbb{R}/\mathbb{Z} \to \mathbb{R}^2.$

First variation

$$\frac{d}{d\varepsilon}E(\gamma + \varepsilon\varphi)\Big|_{\varepsilon=0} = \int_\gamma \nabla_{L^2_{ds}} E(\gamma)\cdot\varphi\, ds, \;\; \varphi \in C^\infty(S^1, \mathbb{R}^2),$$

with

$$\nabla_{L^2_{ds}} E(\gamma) = \left[-\partial_s^4 k - k^2\partial_s^2 k + \frac{1}{2}k(\partial_s k)^2\right]\nu$$

Known results

[1] Andrews-McCoy-Wheeler-Wheeler (2020)

$$\partial_t \gamma = -\nabla_{L^2_{ds}} E(\gamma)$$

$$\left(= \left[\partial_s^4 k + k^2 \partial_s^2 k - \frac{1}{2} k (\partial_s k)^2 \right] \nu \right)$$

$$\text{(A)} \qquad E(\gamma_0) \ll 1, \quad \omega := \frac{1}{2\pi} \int_{\gamma_0} k_0 \, ds \neq 0$$

$\Rightarrow$ (i) Existence of unique global-in-time sol.

(ii) Convergence to an ω-circle

[2] McCoy-Wheeler-Wu (2022)

$$(L(\gamma(t)) \equiv L(\gamma_0))$$

$$\partial_t \gamma = -\nabla_{L^2_{ds}} E(\gamma) + h(t)\nu,$$

$$h(t) = \frac{1}{2\pi\omega} \int_\gamma \left[-(\partial_s^2 k)^2 + \frac{7}{2}(k \partial_s k)^2 \right] ds$$

$$\text{(A)} \quad E(\gamma_0) \ll 1, \quad \omega := \frac{1}{2\pi} \int_{\gamma_0} k_0 \, ds \neq 0$$

$\Rightarrow$ (i) Existence of unique global-in-time sol.

(ii) Convergence to an ω-circle

Aim Construct a gradient flow for E s.t. the flow converges to an ideal curve without (A).

- L^2_{ds}-gradient flow with length constraint

Let

$$V := \left\{ \eta \in L^2(S^1, \mathbb{R}^2) \mid \int_\gamma k\nu \cdot \eta \, ds = 0 \right\}$$

Then

$$\partial_t \gamma = -\nabla_{L^2_{ds}} E(\gamma) + \lambda(t) k\nu \in V \Rightarrow \lambda(t) = \dots$$

$$\partial_t \gamma = -\nabla_{L^2_{ds}} E(\gamma) + \lambda(t) k \nu,$$

$$\lambda(t) = \frac{\int_\gamma \left[-(\partial_s^2 k)^2 + \frac{7}{2}(k \partial_s k)^2\right] ds}{\int_\gamma k^2 \, ds}.$$

$\Rightarrow$ (i) Existence of unique global-in-time sol.
(ii) Convergence to an ideal curve ?

$$\partial_s = \frac{\partial_u}{|\partial_u \gamma|}$$

$$E(\gamma_0) \ll 1$$

$$\Rightarrow \; C_1 \leq |\partial_u \gamma(u,t)| \leq C_2, \forall (u,t) \in S^1 \times [0, \infty)$$

- L^2-gradient flow with inextensibility condition

Let $\gamma_0 : \mathbb{R}/L\mathbb{Z} \to \mathbb{R}^2$ satisfy $|\gamma_0'(x)| \equiv 1.$

$$\partial_t \gamma = \partial_x^6 \gamma + 2\partial_x^2(|\partial_x^2|^2 \partial_x^2 \gamma) + \partial_x(v \partial_x \gamma)$$
$$+\partial_x \left[\left(-\tfrac{9}{2} |\partial_x^3 \gamma|^2 + 3\partial_x^2(|\partial_x^2 \gamma|^2) + \tfrac{13}{2} |\partial_x^2 \gamma|^4 \right) \partial_x \gamma \right]$$

$$-\partial_x^2 v + |\partial_x^2 \gamma|^2 v = -|\partial_x^4 \gamma|^2 + 4\{\partial_x(|\partial_x \gamma|^2)\}^2$$
$$+\tfrac{13}{2} |\partial_x^2 \gamma|^2 |\partial_x^3 \gamma|^2 - \tfrac{13}{2} |\partial_x^2 \gamma|^6$$

$$\Rightarrow \quad \frac{d}{dt} E(\gamma(\cdot, t)) = -\|\partial_t \gamma\|_{L^2_{dx}}^2, \quad |\partial_x \gamma(x,t)| \equiv 1$$

$$-\nabla_{L^2_{dx}} E(\gamma) \qquad V := \{\eta \mid \partial_x \eta \cdot \partial_x \gamma = 0\}$$

$$\partial_t \gamma = \partial_x^6 \gamma + 2\partial_x^2(|\partial_x^2|^2 \partial_x^2 \gamma) + \partial_x(v \partial_x \gamma)$$

$$+\partial_x \left[\left(-\frac{9}{2}|\partial_x^3 \gamma|^2 + 3\partial_x^2(|\partial_x^2 \gamma|^2) + \frac{13}{2}|\partial_x^2 \gamma|^4 \right) \partial_x \gamma \right]$$

$$-\partial_x^2 v + |\partial_x^2 \gamma|^2 v = -|\partial_x^4 \gamma|^2 + 4\{\partial_x(|\partial_x \gamma|^2)\}^2$$

$$+\frac{13}{2}|\partial_x^2 \gamma|^2 |\partial_x^3 \gamma|^2 - \frac{13}{2}|\partial_x^2 \gamma|^6$$

$$\partial_x \partial_t \gamma \cdot \partial_x \gamma = 0$$

$$|\gamma_0'(x)| \equiv 1 \quad \Rightarrow \quad |\partial_x \gamma(x,t)| \equiv 1$$

$$\gamma_0 : \mathbb{R}/L\mathbb{Z} \to \mathbb{R}^2, \quad |\gamma_0'(x)| \equiv 1$$

$\Rightarrow$ (i) Existence of unique global-in-time sol.
(ii) Convergence to an ideal curve

Closed planar Ideal curves

 $\cdots$ $\exists$ figure eight type

multiply covered circles