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# Convergence of solutions of a one-phase Stefan problem with Neumann boundary data to a self-similar profile

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**Summary:**
We explan how a unique self-similar solution to the one-phase one-dimensional Stefan problem attracts all other solutions. The Neumann boundary condition means energy is pumped into the system.

D.Hilhorst, S.Roscani, P.Rybka, Convergence of solutions of a one-phase Stefan problem with Neumann boundary data to a self-similar profile, *NoDEA* **31**, 56 (2024).

I'll talk about ice melting in a simple one-dimensional one-phase case

$$\begin{array}{lll} u_t = u_{xx} & \{(x,t) : x \in (0, s(t)), t \in (0,T)\} & =: \Omega_T, \\ u(s(t),t) = 0, & \dot{s} = -u_x(s(t),t) & t \in (0,T), \\ u(x,0) = u_0(x), & u_0(x) \geq 0,\ u_0(0) > 0, & x \in (0, s_0). \end{array} \tag{1}$$

We have to specify the condition on the fixed part of the boundary. We set the heat flux to be consitent with the decay of the fundamental solution,

$$-u_x(0,t) = \frac{h}{\sqrt{t+1}}, \quad h > 0. \tag{2}$$

Our goal is to discuss the long time behavior. However, the system (1-2) has no steady states, because the energy is pumped into the system. We expect that solutions to (1-2) converge ot a self-similar solution (sss).

A similar problem arising as a corrosion model was studied by DH, there instead of (2) the Dirichlet data was considered.

Since the Stefan problem is classical there is no need to discuss existence of solutions. This was settled:

**Proposition 1** (A.Friedman 1964, D.Andreucci 2004)

If $u_0 \in \mathrm{Lip}\,([0, s_0])$, then there exists $u$ a unique classical solution to (1-2), i.e. $u \in C^{2,1}(\Omega_T)$ and $u, u_x \in C(\bar{\Omega}_T \setminus [0, s_0] \times \{0\})$, $s \in C^1(0, T)$, $\dot{s} \geq 0$.

**Self-similar solutions**

The parabolic scaling suggests that any sss has the form $(u(x,t), s(t)) = (U(\frac{x}{\sqrt{t+1}}), \omega\sqrt{t+1})$. We denote $\omega\sqrt{t+1}$ by $\sigma(t)$. If we plug this into (1-2), then after setting $\eta = \frac{x}{\sqrt{t+1}}$ we obtain

$$\begin{array}{ll} U_{\eta\eta} + \frac{\eta}{2} U_\eta = 0 & \eta \in (0,\omega), \\ -U_\eta = h, & U(\omega) = 0, \\ \frac{\omega}{2} = -U_\eta(\omega). & \end{array} \tag{3}$$

This problem has a unique solution $U(\eta) = h \int_\eta^\omega e^{-\tau^2/4}\, d\tau$, where $\omega$ is the unique solution of

$$h = \frac{y}{2} e^{y^2/4}.$$

**Convergence**

Here is our main result.

**Theorem 2.** (D.Hilhorst, S.Roscani, PR, 2024)

Let us suppose that $0 \leq u \in \mathrm{Lip}\,(\mathbb{R}_+)$, $u_0(0) > 0$, $s_0 > 0$, $u_0(x) = 0$ for $x > s_0$. Then, the unique solution to the Stefan problem (1-2) converges to $(U, \sigma)$. More precisely,

(1) $\lim_{t\to\infty} \frac{s(t)}{\sqrt{t+1}} = \omega$;

(2) $\lim_{t\to\infty} \sup_{\frac{x}{\sqrt{t+1}}\in[0,\omega]} \left| u(x,t) - U\left(\frac{x}{\sqrt{t+1}}\right)\right| = 0.$

Here is the strategy of the proof.

(1) In general, we don't like unbounded domains, so that we transform the problem to a bounded region. This is done with the help of similarity variables $\eta = \frac{x}{\sqrt{t+1}}$, $\tau = \ln(t+1)$. Hence, the spatial variable becomes bounded and the sss will be transformed into a steady state.

(2) We fit the solution to the transformed system, $(W, b)$ between upper $(\overline{W}, \bar{b})$ and lower $(\underline{W}, \underline{b})$ solutions. They converge monotonically in time.

(3) The limits of $(\overline{W}, \bar{b})$ and $(\underline{W}, \underline{b})$ coincide with the steady state of the transformed system. The comparison principle forces convergence of $(W, b)$.

**The transformed problem**

The resulting problem for $(W(\eta,\tau), b(\tau)) = (u(x,t), s(t))$ is

$$\begin{array}{lll} W_\tau = W_{\eta\eta} + \frac{\eta}{2} W_\eta, & \eta \in (0, b(\tau)), & \tau > 0, \\ -W_\eta(0,\tau) = h, & W(b(\tau),\tau) = 0, & \tau > 0, \\ \dot{b}(\tau) + \frac{b(\tau)}{2} = -W_\eta(b(\tau),\tau), & & \tau > 0, \\ W(\eta, 0) = u_0(\eta), & \eta \in (0, b_0), & b(0) = b_0. \end{array} \tag{4}$$

The advantage of this transformation is that the sss given by (3) becomes a steady state of (4).

**Upper and lower solutions**

We introduce classical lower ($\underline{W}, \underline{b}$) and upper solutions ($\overline{W}, \overline{b}$)

$$
\begin{array}{lll}
W_\tau \overset{\leq}{\geq} W_{\eta\eta} + \frac{\eta}{2} W_\eta, & \eta(0, b(\tau)), & \tau > 0, \\
-W_\eta(0,\tau) \overset{\leq}{\geq} h, & W(b(\tau),\tau) = 0, & \tau > 0, \\
\dot{b}(\tau) + \frac{b(\tau)}{2} \overset{\leq}{\geq} -W_\eta(b(\tau),\tau), & & \tau > 0, \\
W(\eta, 0) \overset{\leq}{\geq} u_0(\eta), & \eta \in (0, b_0), & b(0) \overset{\leq}{\geq} b_0.
\end{array}
$$

## Comparison Principle

**Theorem 3.** Let us suppose that $(\underline{W}, \underline{b})$ is a lower solution while $(\overline{W}, \bar{b})$ is an upper solution. If $\underline{W}(\eta, 0) \leq \overline{W}(\eta, 0)$ and $\underline{b}(0) \leq \bar{b}(0)$, then for all $\tau > 0$ we have

$$\underline{W}(\eta, \tau) \leq \overline{W}(\eta, \tau) \quad \text{and} \quad \underline{b}(\tau) \leq \bar{b}(\tau).$$

We find a time independent upper solution $(W^u, b^u)$, such that e.g. $W^u(\eta) \geq u_0(\eta)$, $b^u(0) \geq b_0$. We may also construct non-negative time independent lower solution $(W^l, b^l)$, such that $W^l(\eta) \geq u_0(\eta)$, $b^l(0) \geq b_0$.

We define solutions to (4)

$$\begin{aligned} &(\underline{W}, \underline{b}) \quad \text{with initial data } (W^l, b^l), \\ &(\overline{W}, \overline{b}) \quad \text{with initial data } (W^u, b^u). \end{aligned}$$

The Comparison Principle implies that not only for all $\tau > 0$

$$\underline{W}(\eta, 0) \leq \underline{W}(\eta, \tau) \leq W(\eta, \tau) \leq \overline{W}(\eta, \tau) \leq \overline{W}(\eta, 0)$$

$$\underline{b}(\tau) \leq b(\tau) \leq \overline{b}(\tau),$$

but also for all $\eta \in (0, b_0)$ functions

$$\tau \mapsto \underline{W}(\eta, \tau) \qquad \text{and} \qquad \tau \mapsto \overline{W}(\eta, \tau).$$

are monotone.

**The main theorem** follows from the following result.

**Theorem 4.** Let $(W, b)$ be the solution to (4) with initial condition $(u_0, b_0)$. If $(W^\infty, b^\infty)$ is the unique steady state of the transformed system (4), then:

(a) $\lim_{\tau\to\infty} b(\tau) = b^\infty$;

(b) $W(\cdot,\tau) \rightrightarrows W^\infty$ on $[0,\beta]$ for all $\beta < b^\infty$ when $\tau \to \infty$;

(c) if $W(\cdot,\tau)$ is extended by zero to $\bar{\mathbb{R}}_+$, then $W(\cdot,\tau) \rightrightarrows W^\infty$ on $[0,\infty)$.

*Proof.* Monotonicity of the interface implies that

$$\lim_{\tau\to\infty} \underline{\mathsf{b}}(\tau) = \underline{\mathsf{b}}^\infty \qquad \lim_{\tau\to\infty} \bar{b}(\tau) = \bar{b}^\infty.$$

Moreover,

$$\lim_{\tau\to\infty} \overline{W}(\eta,\tau) = \overline{W}(\eta), \quad \forall \eta \in [0, \bar{b}^\infty],$$

$$\lim_{\tau\to\infty} \underline{\mathsf{W}}(\eta,\tau) = \underline{\mathsf{W}}(\eta), \quad \forall \eta \in [0,\beta], \quad \forall \beta < \underline{\mathsf{b}}^\infty.$$

**Weak convergence in $H^1$**
We have to show that the convergence is uniform and $\overline{W} = \underline{\mathrm{W}}$, $\bar{b}^\infty = \underline{\mathrm{b}}^\infty$ and this it is the only steady state.

We test the equation with the solution $W = \overline{W}$ or $W = \underline{\mathrm{W}}$, formally we have to first multiply $W$ by a cut-off function. We integrate over

$$\Omega_{T,1} = \{(\eta,\tau) \in \mathbb{R}^2_+ : x \in (0, b(\tau)),\ t \in (T, T+1)\}$$

to get

$$L(T) = \int_{\Omega_{T,1}} W W_\tau \, d\eta d\tau = \int_{\Omega_{T,1}} W (W_{\eta\eta} + \frac{\eta}{2} W_\eta W) \, d\eta d\tau = R(T).$$

We integrate by parts to get

$$\begin{aligned} L(T) &= \frac{1}{2}\int_{\Omega_T} \frac{\partial}{\partial\tau} W^2 \, d\eta d\tau = \frac{1}{2}\int_T^{T+1} \frac{\partial}{\partial\tau}\int_0^{b(\tau)} W^2 \, d\eta d\tau \\ &= \frac{1}{2}\int_0^{b(T+1)} W^2 \, d\eta - \frac{1}{2}\int_0^{b(T)} W^2 \, d\eta. \end{aligned}$$

We take care of $R(T)$,

$$
\begin{aligned}
R(T) = \; & -\int_{\Omega_T} W_\eta^2 + \int_T^{T+1} W W_\eta \Big|_{\eta=0}^{\eta=b(\tau)} \, d\tau \\
& -\tfrac{1}{4}\int_{\Omega_T} W^2 \frac{\partial \eta}{\partial \eta} + \int_T^{T+1} \frac{\eta}{2} W^2 \big|_{\eta=0}^{\eta=b(\tau)}.
\end{aligned}
$$

After rearranging the terms and combining with $L(T)$ we obtain

$$
\begin{aligned}
& \frac{1}{2}\int_0^{b(T+1)} W^2(\eta, T+1)\, d\eta + \int_{\Omega_T} (W_\eta^2 + \frac{1}{4} W^2) \\
= \; & \textstyle\int_T^{T+1} W(0,\tau) h + \frac{1}{2}\int_0^{b(T)} W^2(\eta, T)\, d\eta \le D.
\end{aligned}
$$

Thus, the set

$$
\{\tau \in (T, T+1) : \int_0^{b(T+1)} W_\eta^2 \, d\eta \le D\}
$$

has positive measure. Hence, we deduce that $\underline{\mathrm{W}}^\infty \in H^1(0, \underline{\mathrm{b}}^\infty)$ and $\overline{W}^\infty \in H^1(0, \overline{b}^\infty)$.

**Limit identification**

Once we showed existence of the limits $\underline{W}^\infty$ and $\overline{W}^\infty$ we may identify them. We test

$$W_\tau = W_{\eta\eta} + \frac{\eta}{2} W_\eta$$

with $\varphi \in C_c^\infty(\mathbb{R})$, with $\varphi_\eta(0) = 0$,

$$\begin{aligned} L_2(T) &= \int_T^{T+1} \int_0^{b(\tau)} W_\tau(\eta,\tau)\varphi(\eta)\, d\eta d\tau \\ &= \int_T^{T+1} \frac{\partial}{\partial \tau} \int_0^{b(\tau)} W(\eta,\tau)\varphi(\eta)\, d\eta d\tau \\ &= \int_0^{b(T+1)} W(\eta, T+1)\varphi(\eta)\, d\eta - \int_0^{b(T)} W(\eta,T)\varphi(\eta)\, d\eta. \end{aligned}$$

Obviously, $\lim_{T\to\infty} L_2(T) = 0$.

$$
\begin{aligned}
R_2(T) &= -\int_{\Omega_{T,1}} W_\eta(\eta,\tau)\varphi_\eta(\eta)\,d\eta d\tau + \int_T^{T+1} W_\eta \varphi|_{\eta=0}^{\eta=b(\tau)} \\
&\quad -\int_{\Omega_{T,1}} \frac{1}{2}(\eta\varphi(\eta))_\eta W(\eta,\tau)\,d\eta d\tau + \int_T^{T+1} \frac{\eta}{2} W\varphi(\eta)|_{\eta=0}^{\eta=b(\tau)}\,d\tau \\
&= \int_{\Omega_{T,1}} W(\eta,\tau)\varphi_{\eta\eta}(\eta)\,d\eta d\tau - \int_T^{T+1} W\varphi_\eta|_{\eta=0}^{\eta=b(\tau)}\,d\tau \\
&\quad +\int_T^{T+1}((-\dot{b}-\frac{b}{2})\varphi(b)+h\varphi(0))\,d\tau \\
&\quad -\int_{\Omega_{T,1}} \frac{1}{2}(\eta\varphi(\eta))_\eta W(\eta,\tau)\,d\eta d\tau + 0 \\
&= \int_{\Omega_{T,1}} W(\varphi_{\eta\eta} - (\frac{\eta}{2}\varphi)_\eta)\,d\eta\tau + h\varphi(0) \\
&\quad +\int_T^{T+1}(W(0,\tau)\varphi_\eta(0) - (\dot{b}-\frac{b}{2})\varphi(b))\,d\tau
\end{aligned}
$$

We see $0 = \lim_{T\to\infty} L_2(T) = \lim_{T\to\infty} R_2(T)$ and for $U = \underline{W}^\infty$ or $U = \overline{W}^\infty$

$$\lim_{T\to\infty} R_2(T) = \int_0^{b^\infty} U(\eta)(\varphi_{\eta\eta} - (\frac{\eta}{2}\varphi)_\eta) \\ +h\varphi(0) + U(0)\varphi_\eta(0) + \frac{b}{2}\varphi(b),$$

because

$$\lim_{T\to\infty} (\Phi(b(T+1)) - \Phi(b(T))) = 0,$$

where $\Phi$ is any antiderivative of $\varphi$, $\Phi' = \varphi$. Finally, $U$ and $a = \underline{b}^\infty$ or $a = \overline{b}^\infty$ satisfy

$$0 = \int_0^a U(\eta)(\varphi_{\eta\eta} - (\frac{\eta}{2}\varphi)_\eta)\, d\eta + h\varphi(0) + U(0)\varphi_\eta(0) + \frac{a}{2}\varphi(a),$$

i.e. the couple $(U, a)$ is a distrubutional solution to the steady state eq. Since we know that $U \in H^1(0, a)$, we deduce that $U \in H^2(0, a)$ and $U$ satisfies the boundary conditions,

$$U_\eta(0) = -h, \quad U(a) = 0, \quad U_\eta(a) = -\frac{a}{2}.$$

This means that $(U, a)$ is a unique steady state $(W^\infty, b^\infty)$, hence

$$(\underline{W}^\infty, \underline{b}^\infty) = (U, a) = (\overline{W}^\infty, \overline{b}^\infty).$$

Now, we improve the convergence. Since $W^\infty \in H^1(0, b^\infty)$ we infer that $W^\infty \in C^{1/2}((0, b^\infty])$. Since the sequences $\underline{W}$ and $\overline{W}$ are monotone and converge to a continuous function we deduce by Dini Theorem, that the convergence is uniform on $[0, b^\infty]$, as desired.

After returning to the original variable we conclude the proof of our theorem.