A combined shape and topology optimisation using phase fields and the $W^{1,\infty}$ topology

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UNIVERSITY OF SUSSEX

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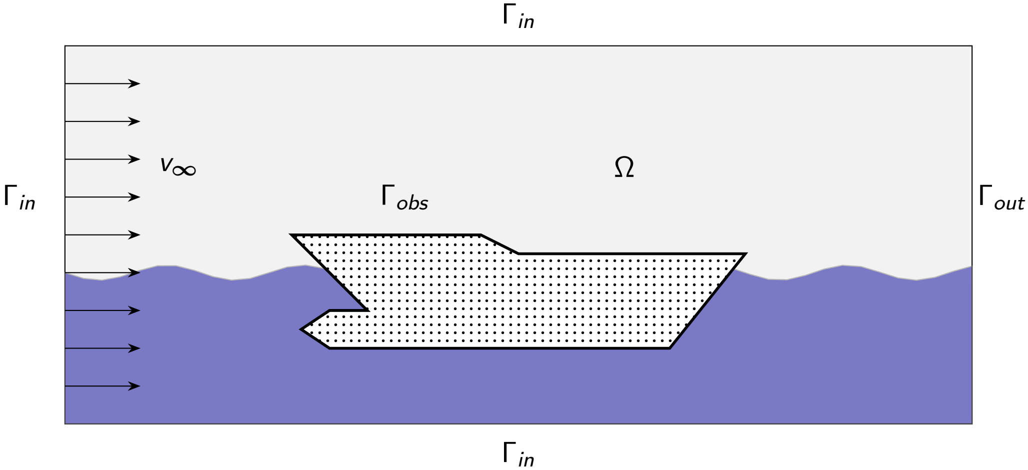
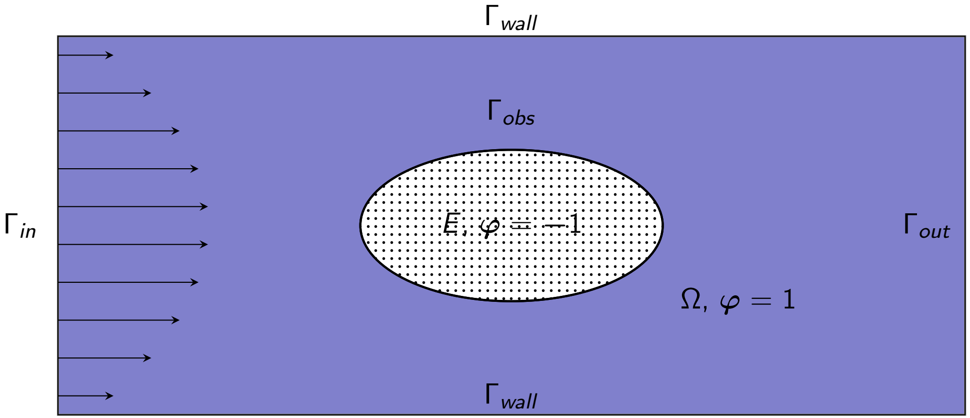
Motivation

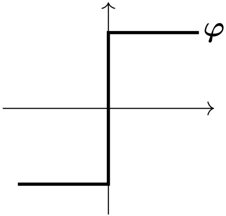


Motivation simplified



Two approaches

Sharp interface $\varphi \in BV(D; \{-1, 1\})$



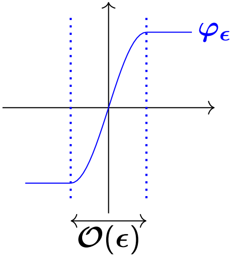
Positives:

- Corners can be quite natural.
- Solving the PDE.

Negatives:

- Have to move the mesh.
- Mesh can degenerate; requires remeshing.
- Topology changes are challenging.

Diffuse interface $\varphi_\epsilon \approx \varphi$, $\varphi_\epsilon \in H^1(D; [-1, 1])$



Positives:

- Only have to change the density.
- Standard methods for refinement.
- Significant amount of developed literature.

Negatives:

- Solving an approximation.
- Parameter tuning.

Two approaches to the same problem

Sharp interface

Let $\Omega := \{\varphi > 0\}$, $E := \{\varphi < 0\}$, and $\Gamma := \partial\Omega$.

$$\min_{y \in U^{\varphi}, \varphi \in \Phi_{ad}} J(y, \varphi)$$

subject to

$$-\mu \Delta y_{vel} + (y_{vel} \cdot \nabla) y_{vel} + \nabla y_{press} = 0 \text{ in } \Omega$$
$$\operatorname{div} y_{vel} = 0 \text{ in } \Omega$$
$$y_{vel} = g \text{ on } \partial\Omega \cap \partial D$$
$$y_{vel} = 0 \text{ on } \partial\Omega \setminus \partial D$$

where, e.g.,
$\Phi_{ad} = \{\varphi \in BV(D; \{-1, 1\}) : \int_D \varphi \,\mathrm{dx} = \beta|D|\}$ and $U^{\varphi} = \{y \in H^1 \times L^2 : y_{vel} = 0 \text{ on } \varphi = -1, y_{vel}|_{\partial D} = g\}$

Diffuse interface

Let $\Omega_\epsilon := \{\varphi_\epsilon = 1\}$, $E_\epsilon := \{\varphi = -1\}$, and $\Gamma_\epsilon := \{|\varphi_\epsilon| < 1\}$.

$$\min_{y \in U, \varphi \in \Phi_{ad,\epsilon}} J(y, \varphi_\epsilon) + \frac{\gamma}{c_0} \int_D \left(\epsilon |\nabla \varphi_\epsilon|^2 + W(\varphi_\epsilon) \right) \mathrm{dx} + \int_\Omega \frac{1}{2} \alpha_\epsilon(\varphi_\epsilon) |y_{vel}|^2 \mathrm{dx}$$

subject to

$$-\mu \Delta y_{vel} + (y_{vel} \cdot \nabla) y_{vel} + \alpha_\epsilon(\varphi_\epsilon) y_{vel} + \nabla y_{press} = 0 \text{ in } \Omega$$
$$\operatorname{div} y_{vel} = 0 \text{ in } \Omega$$
$$y_{vel} = g \text{ on } \partial D$$

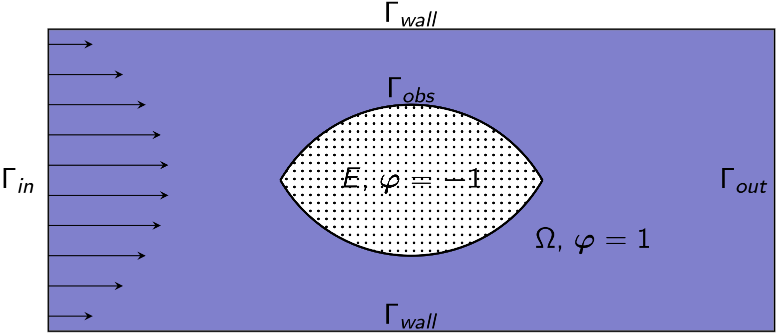
where, e.g.,
$\Phi_{ad} = \{\varphi \in H^1(D; [-1, 1]) : \int_D \varphi \,\mathrm{dx} = \beta|D|\}$ and
$U^{\varphi} = \{y \in H^1 \times L^2 : y_{vel}|_{\partial D} = g\}$

A possible energy function

One typical example for J is to choose

$$J(y,\varphi) := \int_D \frac{1+\varphi}{2}\frac{\mu}{2}|Dy_{vel}|^2 \mathrm{dx}.$$

This is expected to have a minimiser of the form



Computational shape optimisation with phase fields

Domain D is discretised by a triangulation $\mathcal{T}_h$, with U being discretised by Taylor-Hood elements subordinate to $\mathcal{T}_h$. Piecewise linear functions are used to discretise $\Phi_{ad,\epsilon}$.
The minimisation process uses VMPT[1]:

$$\varphi_\epsilon^{k+1} = (1 - t_k)\varphi_\epsilon^k + t_k \hat{\varphi}_\epsilon^{k+1}$$

where $t_k \in (0, 1]$ is a step-size and

$$\hat{\varphi}_\epsilon^{k+1} = \arg\min \left\{ \frac{1}{2} \|\phi - \varphi_\epsilon^k\|_H^2 + j'(\varphi_\epsilon^k)[\phi - \varphi_\epsilon^k] : \phi \in \Phi_{ad,\epsilon} \right\},$$

where j_ϵ is the reduced cost functional and $H = H^1(D)$.

[1] L. Blank and C. Rupprecht. "An extension of the projected gradient method to a Banach space setting with application in structural topology optimization". In: *SIAM Journal on Control and Optimization* 55.3 (2017), pp. 1481–1499.

Computational shape optimisation with sharp interface

This is more challenging. Recent typical methods involve Hilbert spaces, e.g., $H^1(\Omega;\mathbb{R}^d)$.
Writing $j(\Omega) = J(y,\varphi)$ for the reduced (shape) functional, the shape derivative

$$j'(\Omega)[V] := \lim_{t\to 0^+} \frac{j((\mathrm{id}+tV)(\Omega)) - j(\Omega)}{t}$$

is only generally defined for $V \in W^{1,\infty}(\Omega)$.
There are a few works[2][3][4] which use the approach of minimsing using the $W^{1,\infty}$ topology; furthermore, they do not (yet) handle geometric constraints unless using a penalty.

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[2] K. Deckelnick, P. J. Herbert, and M. Hinze. "A novel $W^{1,\infty}$ approach to shape optimisation with Lipschitz domains". In: *ESAIM: COCV* 28 (2022).

[3] K. Deckelnick, P. J. Herbert, and M. Hinze. *Convergence of a steepest descent algorithm in shape optimisation using $W^{1,\infty}$ functions*. (under revision).

[4] K. Deckelnick, P. J. Herbert, and M. Hinze. *PDE constrained shape optimisation with first-order and Newton-type methods in the $W^{1,\infty}$ topology*. (under revision).

Computational shape optimisation with sharp interface (geometric penalty)

We saw some of this on Monday, so this is only a quick refresher

The domain D is discretised by a triangulation $\mathcal{T}_{\Phi_h^0}$. The initial guess $\hat{\Omega}$ should be a collection of these triangles. Our mesh will be parameterised according to a piecewise linear function Φ_h^n. The domain $\hat{\Omega}$ is triangulated by $\mathcal{T}_{\Omega_{\Phi_h^0}}$, a sub triangulation of $\mathcal{T}_{\Phi_h^0}$. Here, $\Omega_{\Phi_h^n}$, denotes $\Phi_h^n(\hat{\Omega})$.

On $\mathcal{T}_{\Omega_{\Phi_h^n}}$, we use Taylor-Hood elements to discretise U^{φ}.

The entire computational mesh is updated according to

$$\Phi_h^{n+1} = (\mathrm{id} + t_k V_h^n) \circ \Phi_h^n$$

where $t_k \in (0, 1)$ and

$$V_h^n \in \arg\min\{j'_\delta(\Omega_{\Phi_h^n})[V_h] : V_h \in \mathcal{V}_{\Phi_h^n}, |DV_h| \leq 1\},$$

where $j_\delta = j + \frac{1}{\delta}$(penalty term), and $\mathcal{V}_{\Phi_h^h}$ are piecewise linear functions on D subordinate to $\mathcal{T}_{\Phi_h^n}$.

For a Poisson state problem, we have global convergence of this method. With assumptions, it is known that $j'_{\delta_h}(\Omega_{\Phi_h^n}) \to 0$.

Computational shape optimisation with sharp interface (geometric constraint)

Not so much is different if one wishes to incorporate the geometric constraint $G(\Omega) = 0$, only the update step, whereby one takes

$$V_h^n \in \arg\min\{j'(\Omega_{\Phi_h^n})[V_h] : V_h \in \mathcal{V}_{\Phi_h^n}, |DV_h| \leq 1, G((\mathrm{id} + tV_h)(\Omega_{\Phi_h^n})) = 0\}.$$

We can still run the code for this, albeit without a convergence argument[5].
An advantage of this is that one does not have to deal with a penalty parameter, its tuning, and the slowness introduced. A disadvantage is the dependence of V_h^n on t, as well as a more difficult problem to solve.

[5] we are working on this

Our proposed strategy

1. Use a phase field method to find a diffuse interface (almost minimiser).
2. Make cuts in the triangulation along the zero level set.
3. Do the sharp method.
4. (No reason one couldn't go back to the diffuse approach and iterate)

Experiment

Experiments conducted using DUNE.

http://dune-project.org/

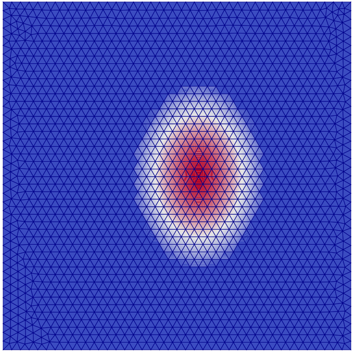
We consider three problems:

- A simple Poisson problem for a kidney shape - a common example in the shape optimisation literature;
- The Stokes problem with constrained barycenter and volume, with the energy being the viscous dissipation;
- The Stationary Navier–Stokes problem with constrained barycenter and volume, with the energy being the viscous dissipation.

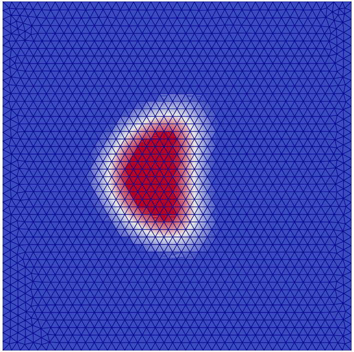
Simple (Poisson) problem

Kidney shape

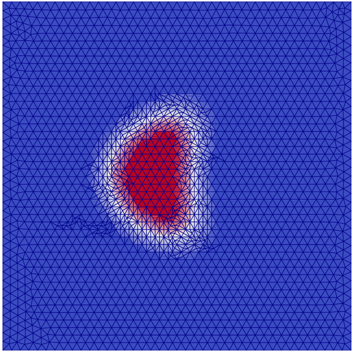
We consider $j(\Omega) = \int_\Omega y \, \mathrm{dx}$, where $y \in H_0^1(\Omega;\mathbb{R})$ satisfies $-\Delta y = F$,
$F = 10(2.5(x_1 + 0.5 - x_2^2)^2 + x_1^2 + x_2^2 - 1)$.



Initial phase



Phase before 1st refinement

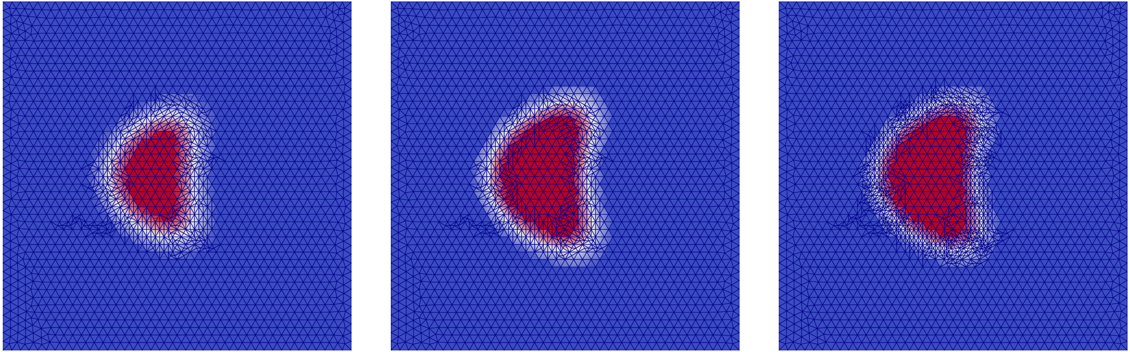


Phase after 1st refinement

Simple (Poisson) problem

Kidney shape

We consider $j(\Omega) = \int_\Omega y \, \mathrm{d}x$, where $y \in H^1_0(\Omega; \mathbb{R})$ satisfies $-\Delta y = F$, $F = 10(2.5(x_1 + 0.5 - x_2^2)^2 + x_1^2 + x_2^2 - 1)$.



Phase after 1st refinement | Phase before 2nd refinement | Phase after 2nd refinement

Simple (Poisson) problem

Kidney shape

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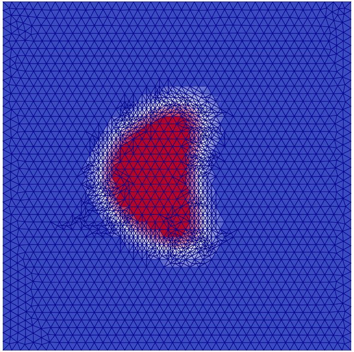
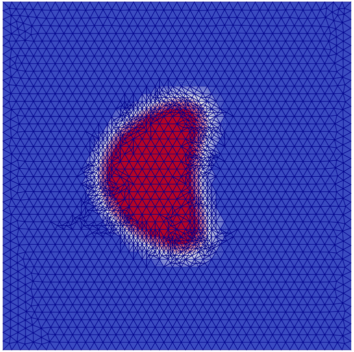
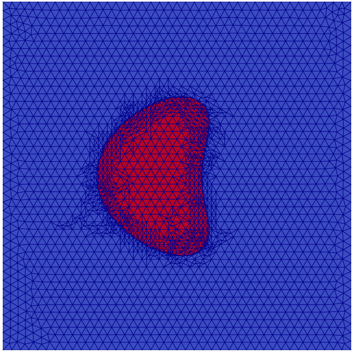


Phase after 2nd refinement



Phase before sharp method



Initial sharp phase

Simple (Poisson) problem

Kidney shape

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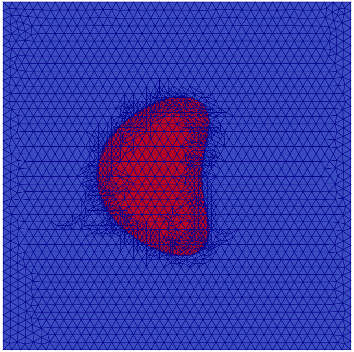
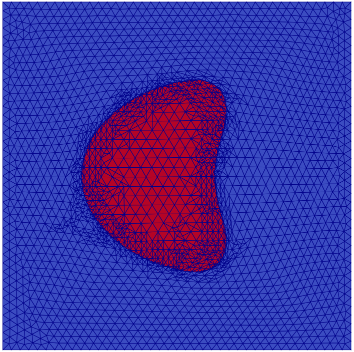


Initial sharp phase



Final sharp phase

Stokes problem

We consider $j(\Omega) = \frac{\mu}{2} \int_\Omega |Dy_{vel}|^2 \,\mathrm{dx}$, where $y = (y_{vel}, y_{press})$ solves the Stokes equations.

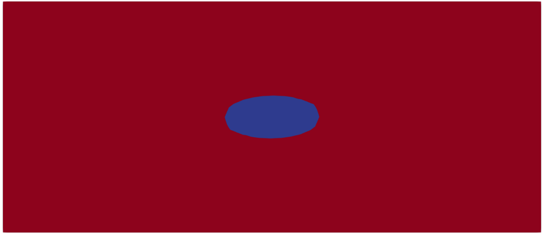
Initial phase

Phase before sharp method

Stokes problem

We consider $j(\Omega) = \frac{\mu}{2} \int_\Omega |Dy_{vel}|^2 \, \mathrm{d}x$, where $y = (y_{vel}, y_{press})$ solves the Stokes equations.

Phase before sharp method



Initial sharp phase

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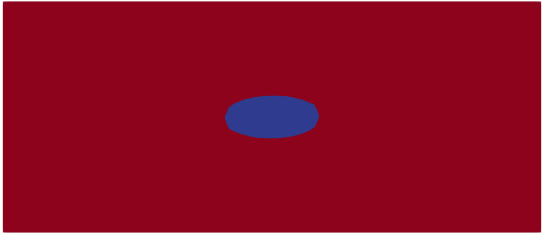
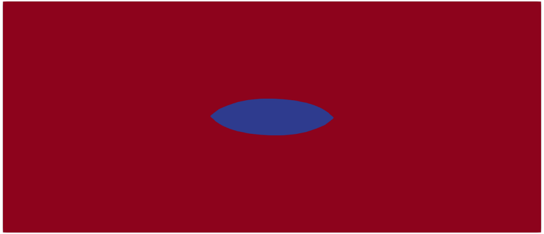


Initial sharp phase



Final sharp phase

Navier–Stokes problem

We consider $j(\Omega) = \frac{\mu}{2}\int_\Omega |Dy_{vel}|^2 \,\mathrm{dx}$, where $y = (y_{vel}, y_{press})$ solves the stationary-Navier–Stokes equations.

Initial phase

Phase before sharp method

Navier–Stokes problem

We consider $j(\Omega) = \frac{\mu}{2} \int_\Omega |Dy_{vel}|^2 \, \mathrm{dx}$, where $y = (y_{vel}, y_{press})$ solves the stationary-Navier–Stokes equations.

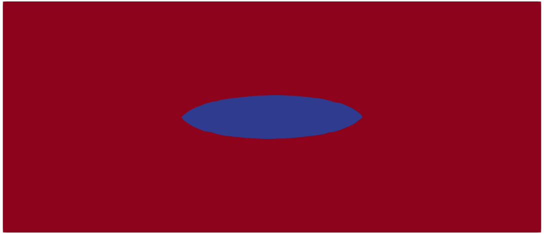
Phase before sharp method

Initial sharp phase

Navier–Stokes problem

We consider $j(\Omega) = \frac{\mu}{2} \int_\Omega |Dy_{vel}|^2 \,\mathrm{dx}$, where $y = (y_{vel}, y_{press})$ solves the stationary-Navier–Stokes equations.

Initial sharp phase



Final sharp phase

Summary

We propose a shape optimisation framework together with a steepest descent method for shape optimisation in the $W^{1,\infty}$ topology.

We have shown:

- Seen the transerfence of a phase field problem to a sharp interface problem for shape optimisation for a non-trivial example.
- Seen that the phase field gives a *good guess*, and that the sharp method makes corners where they should be present.

Future work:

- Demonstrate convergence with geometric constraints in the sharp problem.
- Higher order (second derivative) information.
- Utilise this combined approach for problems with many topology changes, e.g., elasticity.

Bibliography

Thank you for your attention!

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