Existence for a class of fourth-order quasilinear parabolic equations

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joint work with Y. Giga

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Motivation

Macroscopic models of thermodynamic fluctuations of crystal surfaces

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\begin{aligned}
u_t &= \operatorname{div}\left(\mathbb{M}(\nabla u)\nabla f(-\operatorname{div} D\Phi(\nabla u))\right) \\
&= -\operatorname{div}\big(\mathbb{M}(\nabla u) f'(-\operatorname{div} D\Phi(\nabla u))\nabla \operatorname{div} D\Phi(\nabla u)\big)
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physical choices:

- $f(p) = e^p$, $f(p) = p$
- $\Phi(\boldsymbol{\xi}) \sim |\boldsymbol{\xi}|^p, p > 1$ (Marzuola-Weare 2013)
- $\Phi(\boldsymbol{\xi}) \sim |\boldsymbol{\xi}|$ (Liu-Lu-Margetis-Marzuola 2017)
- $\Phi(\boldsymbol{\xi}) \sim |\boldsymbol{\xi}| + |\boldsymbol{\xi}|^3$ (Margetis-Kohn 2006)
- $\mathbb{M}(\boldsymbol{\xi}) \sim \begin{bmatrix} 1 & 0 \\ 0 & (1+|\boldsymbol{\xi}|)^{-1} \end{bmatrix}$, $\mathbb{M}(\boldsymbol{\xi}) \sim \mathbb{U}(\boldsymbol{\xi})^T \begin{bmatrix} 1 & 0 \\ 0 & (1+|\boldsymbol{\xi}|)^{-1} \end{bmatrix} \mathbb{U}(\boldsymbol{\xi})$ (Margetis-Kohn 2006)

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- numerics for discretization of $u_t = -\mathrm{div} e^{-\psi_\varepsilon * \Delta_1 u} \nabla \Delta_1 u$, stability of spatial discretization (Craig-Liu-Lu-Marzuola-Wang, 2022)
- existence for $u_t = \Delta \exp(-\Delta_p u)$, $1 < p \leq 2$ (Price-Xu, 2023)

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- existence results rely on gradient flow formulations or monotonicity

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- $\mathcal{F}(u) = \int_\Omega \exp(-(\ln u_x)_x)$ (Gao 2019)

Monotone operators

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$$\langle \mathcal{L}(v) - \mathcal{L}(w), v - w \rangle \leq 0$$

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General existence result using Galerkin method by Vishik (1962) for parabolic systems of form

$$\boldsymbol{u}_t + (-1)^n \mathrm{div}^n \, \mathbb{L}(t, x, \boldsymbol{u}, \nabla \boldsymbol{u}, \ldots, \nabla^n \boldsymbol{u}) = \boldsymbol{g}(t, x)$$

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Example:

- $\Phi(\boldsymbol{\xi}) = |\boldsymbol{\xi}|$, $u_t = -\Delta \mathrm{div} \frac{\nabla u}{|\nabla u|}$ (Giga-Giga 2010, Giga-Kuroda-Matsuoka 2014, Giga-Kuroda-Ł 2023)

Weighted H^{-1} gradient flows

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Damlamian, 1974: existence of $H^{-1}_{\mathbb{B}}$ gradient flows assuming

- $t \mapsto \mathbb{B}(t,\cdot)$ is a family of uniformly equivalent scalar products on $\mathbb{R}^n$
- $\mathbb{B} \in W^{1,1}(0,T,L^\infty(\Omega))$

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- $u_t = -\mathrm{div}\, b(u)\nabla(\Delta u - g(u))$ (Lisini-Matthes-Savaré 2012)
- $\mathrm{div}\, \mathbb{B}(\boldsymbol{u})\nabla$ via Wasserstein metrics (Mielke, 2011; Liero-Mielke 2013)

Basic result: assumptions

$$\boldsymbol{u}_t + \operatorname{div} \mathcal{B}(\boldsymbol{u}) \nabla \mathcal{A}(\boldsymbol{u}) = \operatorname{div} \boldsymbol{g} \tag{$*$}$$

- for simplicity $\Omega = \mathbb{T}^n$
- $\mathcal{A}(\boldsymbol{u}) = \operatorname{div} D_{\boldsymbol{\xi}} \Phi(x, \nabla \boldsymbol{u})$
- Φ is convex and C^1 with respect to the gradient variable,

  $$c_0(|\boldsymbol{\xi}|^p - 1) \leq \Phi(x, \boldsymbol{\xi}), \qquad |D_{\boldsymbol{\xi}} \Phi(x, \boldsymbol{\xi})| \leq c_1(|\boldsymbol{\xi}|^{p-1} + 1)$$

  with $p > \max(1, \frac{2n}{n+4})$
- $\mathcal{B}(\boldsymbol{u}) = \mathbb{B}(t, x, \boldsymbol{u}, \nabla \boldsymbol{u}, \mathcal{A}(\boldsymbol{u}))$ with a Carathéodory function $\mathbb{B}$ taking values in $L(\mathbb{R}^{nN}, \mathbb{R}^{nN})$ satisfying

  $$\mu \mathbb{I} \leq \mathbb{B} \leq M \mathbb{I}$$

- $\boldsymbol{g} = \boldsymbol{g}(t, x) \in L^2(]0, T[\times \Omega)^{nN}$

Basic result: statement

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Theorem (Giga-Ł, in preparation)

Let $\boldsymbol{u}_0 \in W^{1,p}(\Omega)$ and let $T > 0$. There exists a weak solution to $(*)$ in $]0, T[\times\Omega$ with initial datum $\boldsymbol{u}_0$ satisfying energy inequality

$$\sup_{0<t<T} \int_\Omega \Phi(\cdot, \nabla \boldsymbol{u}) + \frac{1}{2} \int_0^T \int_\Omega \nabla \mathcal{A}(\boldsymbol{u}) : \mathcal{B}(\boldsymbol{u}) \nabla \mathcal{A}(\boldsymbol{u}) \leq 2 \int_\Omega \Phi(\cdot, \nabla \boldsymbol{u}_0) + \frac{1}{\mu} \int_0^T \int_\Omega |\boldsymbol{g}|^2.$$

Sketch of proof

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$$\boldsymbol{u}_{j,t} + \mathrm{div}\, \mathcal{P}_j \left(\mathcal{B}_j(\boldsymbol{u}_j) \nabla \mathcal{A}_j(\boldsymbol{u}_j) \right) = \mathrm{div}\, \mathcal{P}_j\, \boldsymbol{g},$$

where $\mathcal{P}_j$ – projection onto first j eigenvectors and

$$\mathcal{A}_j(\boldsymbol{w}) = \mathrm{div}\, \mathcal{P}_j\, D_{\boldsymbol{\xi}} \Phi(x, \nabla \boldsymbol{w}), \quad \mathcal{B}_j(\boldsymbol{w}) = \mathbb{B}(t, x, \boldsymbol{w}, \nabla \boldsymbol{w}, \mathcal{A}_j(\boldsymbol{w}))$$

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  satisfy energy inequality
- by boundedness of energy and monotonicity we have strong convergence $\nabla\boldsymbol{u}_j \to \nabla\boldsymbol{u}$, $D_{\boldsymbol{\xi}}\Phi(\cdot,\nabla\boldsymbol{u}_j) \to D_{\boldsymbol{\xi}}\Phi(\cdot,\nabla\boldsymbol{u})$

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- using the L^2 bound on $\nabla\mathcal{A}_j(\boldsymbol{w})$ and interpolation, we obtain strong convergence of $\mathcal{A}_j(\boldsymbol{u}_j)$ and then $\mathcal{B}_j(\boldsymbol{u}_j)$

Galerkin approximation

$\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \ldots$ — orthogonal eigenbasis of Δ on $L^2_{av}(\Omega)^N$

$\nabla\boldsymbol{\omega}_1, \nabla\boldsymbol{\omega}_2, \ldots$ — orthogonal eigenbasis of $\nabla\mathrm{div} = \Delta$ on $L^2_{\nabla}(\Omega)^{nN}$

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$$\boldsymbol{u}_{j,t} + \operatorname{div}\mathcal{P}_j\,(\mathcal{B}_j(\boldsymbol{u}_j)\nabla\mathcal{A}_j(\boldsymbol{u}_j)) = \operatorname{div}\mathcal{P}_j\,\boldsymbol{g},$$

$\boldsymbol{u}_j(0, \cdot) = \boldsymbol{u}_{0,j}$ — projection onto $\operatorname{span}(\boldsymbol{e}_1, \dots, \boldsymbol{e}_N, \boldsymbol{\omega}_1, \dots, \boldsymbol{\omega}_j)$.

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$$\boldsymbol{u}_j(t,\cdot) = (a_1, \dots, a_N) + \sum_{i=1}^{j} a_{ji}(t)\boldsymbol{\omega}_i,$$

Approximate energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \Phi(\cdot, \nabla \boldsymbol{u}_j) = \int_\Omega D_{\boldsymbol{\xi}}\Phi(\cdot, \nabla \boldsymbol{u}_j) : \nabla \boldsymbol{u}_{j,t}$$

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$$\sup_{0<t<T}\int_\Omega \Phi(\cdot,\nabla \boldsymbol{u}_j) + \frac{1}{2}\int_0^T\!\!\int_\Omega \nabla \mathcal{A}_j(\boldsymbol{u}_j)\cdot \mathcal{B}_j(\boldsymbol{u}_j)\nabla \mathcal{A}_j(\boldsymbol{u}_j)$$

$$\leq 2\int_\Omega \Phi(\cdot,\nabla \boldsymbol{u}_{0,j}) + \frac{1}{\mu}\int_0^T\!\!\int_\Omega |\boldsymbol{g}|^2.$$

Limit passage I

$\boldsymbol{u}_j$ — uniformly bounded in $L^\infty(0,T,W^{1,p}(\Omega))$

$\mathcal{A}_j(\boldsymbol{u}_j)$ — uniformly bounded in $L^2(0,T,H^1(\Omega))$

$\boldsymbol{u}_{j,t} = -\operatorname{div}\mathcal{P}_j\left(\mathcal{B}_j(\boldsymbol{u}_j)\nabla\mathcal{A}_j(\boldsymbol{u}_j)\right) + \operatorname{div}\mathcal{P}_j\,\boldsymbol{g}$

$\boldsymbol{u}_{j,t}$ — uniformly bounded in $L^2(0,T,H^{-1}(\Omega))$

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Mazur's lemma, integration by parts

$$\Longrightarrow\ \lim_{j\to\infty}\int_0^T\int_\Omega (D\Phi(\cdot,\nabla\boldsymbol{u}_j) - D\Phi(\cdot,\nabla\boldsymbol{u}))\cdot(\nabla\boldsymbol{u}_j - \nabla\boldsymbol{u}) = 0$$

Limit passage I

$\boldsymbol{u}_j$ — uniformly bounded in $L^\infty(0, T, W^{1,p}(\Omega))$
$\mathcal{A}_j(\boldsymbol{u}_j)$ — uniformly bounded in $L^2(0, T, H^1(\Omega))$
$\boldsymbol{u}_{j,t} = -\operatorname{div} \mathcal{P}_j \left(\mathcal{B}_j(\boldsymbol{u}_j) \nabla \mathcal{A}_j(\boldsymbol{u}_j) \right) + \operatorname{div} \mathcal{P}_j \, \boldsymbol{g}$
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$$\implies \boldsymbol{u}_j \to \boldsymbol{u} \text{ in } C([0,T], L^q(\Omega)), \quad q < p^*$$

Mazur's lemma, integration by parts

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$$\implies D_{\boldsymbol{\xi}}\Phi(\cdot,\nabla \boldsymbol{u}_j) \to D_{\boldsymbol{\xi}}\Phi(\cdot,\nabla \boldsymbol{u}) \text{ in } L^s(0,T,L^{r'}(\Omega)), \quad r' < p'$$

Limit passage II

$$\int_0^T \int_\Omega (\mathcal{A}_j(\boldsymbol{u}_j) - \mathcal{A}(\boldsymbol{u}))^2$$

$$= - \int_0^T \int_\Omega (\nabla \mathcal{A}_j(\boldsymbol{u}_j) - \nabla \mathcal{A}(\boldsymbol{u})) \cdot (\mathcal{P}_j D_{\boldsymbol{\xi}} \Phi(\cdot, \nabla \boldsymbol{u}_j) - \mathcal{P}_\nabla D_{\boldsymbol{\xi}} \Phi(\cdot, \nabla \boldsymbol{u}))$$

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$$\begin{aligned} \mathcal{P}_j D_{\boldsymbol{\xi}} \Phi(\cdot, \nabla \boldsymbol{u}_j) - \mathcal{P}_\nabla D_{\boldsymbol{\xi}} \Phi(\cdot, \nabla \boldsymbol{u}) & \\ &= (\mathcal{P}_j D_{\boldsymbol{\xi}} \Phi(\cdot, \nabla \boldsymbol{u}_j) - \mathcal{P}_j D_{\boldsymbol{\xi}} \Phi(\cdot, \nabla \boldsymbol{u})) \\ &+ (\mathcal{P}_j D_{\boldsymbol{\xi}} \Phi(\cdot, \nabla \boldsymbol{u}) - \mathcal{P}_\nabla D_{\boldsymbol{\xi}} \Phi(\cdot, \nabla \boldsymbol{u})) \\ &=: \mathcal{P}_j M_j + N_j. \end{aligned}$$

$$N_j \to 0 \text{ in } L^2(0, T, L^2(\Omega))$$

Limit passage III

$$M_j = D_{\boldsymbol{\xi}}\Phi(\cdot, \nabla \boldsymbol{u}_j) - D_{\boldsymbol{\xi}}\Phi(\cdot, \nabla \boldsymbol{u})$$

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Limit passage IV

$$\mathcal{P}_j D_{\boldsymbol{\xi}} \Phi(\cdot, \nabla \boldsymbol{u}_j) \to \mathcal{P}_{\nabla} D_{\boldsymbol{\xi}} \Phi(\cdot, \nabla \boldsymbol{u}) \text{ in } L^2(\Omega_T)$$

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Scope

Generalizations:

- unbounded $\mathbb{B}$

$$|\mathbb{B}(t, x, \boldsymbol{w}, \boldsymbol{\xi}, \boldsymbol{z})| \leq c_2 \left(f_2(t, x) + |\boldsymbol{w}|^{q_0} + |\boldsymbol{\xi}|^{q_1} + |\boldsymbol{z}|^{q_2}\right)$$

  with $f_2 \in L^1(]0, T[\times\Omega)$, $q_0 < p^*$, $q_1 < p$, $q_2 < 2$

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- boundary conditions; we need a mild regularity assumption on Ω (C^1 is enough)

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Thank you for your attention!