Shape optimization with Lipschitz methods[1]

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81. Fujihara Seminar
Mathematical Aspects of Interfaces and Free Boundaries

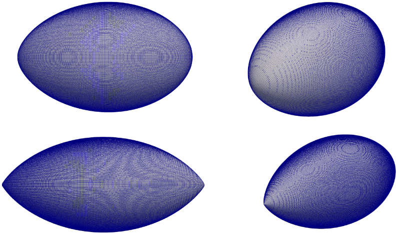
Niseko, June 3, 2024

[1] joint work with Klaus Deckelnick & Philip Herbert

Motivation for $W^{1,\infty}$–topology

$$\min_{\Omega \in \mathcal{S}} \mathcal{J}(\Omega) = \int_\Omega \frac{1}{2}\|y - z\|^2\, dx \text{ s.t.}$$
$$-\Delta y = f \text{ in } \Omega,\ y = 0 \text{ on } \partial\Omega.$$

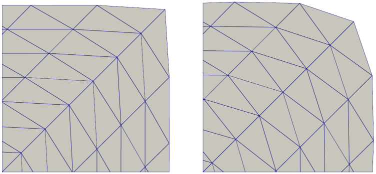
H^1 **(top) versus** $W^{1,p}$ **(bottom)**[a]



How does a shape derivative in PDE constrained shape optimization look like?

$$J'(\Omega)(V) =$$
$$\int_\Omega (DV + DV^t - divV\mathcal{I})\nabla y \nabla p dx +$$
$$+ \int_\Omega \frac{1}{2}(y-z)^2 divV - (y-z)\nabla z V dx -$$
$$- \int_\Omega f V \nabla p dx$$

$W^{1,\infty}$ **(left) versus** H^1 **(right)**[b]



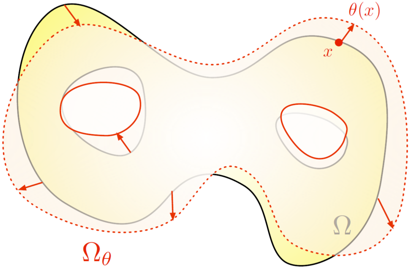
Here, $p \in H_0^1$ **denotes the adjoint variable satisfying** $-\Delta p = (y - z)$ **in** Ω**.**

[a] Numerics by Peter Marvin Müller
[b] Numerics by Philip Herbert

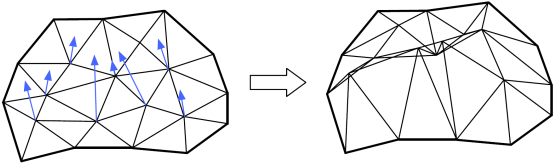
Outline

- **Descent in $W^{1,\infty}$**
- **Hold-all concept**
- **Finite element approximation**
- **Convergence**
- **Summary**

Descent in $W^{1,\infty}$



Deformation of a domain by vector fields θ[2]



Grid quality should be ***conserved***

[2] Courtesy G. Allaire, C. Dapogny, and F. Jouve. Chapter 1 - Shape and topology optimization. In: Geometric Partial Differential Equations - Part II. Vol. 22. Handbook of Numerical Analysis. Elsevier, 2021, pp. 1-132.

Descent in $W^{1,\infty}$

Let $J : \mathcal{A} \to \mathbb{R}$ denote a shape functional, where we assume that J is shape differentiable in an appropriate sense with differential DJ. Let $\Omega \subset \mathbb{R}^d$ be a bounded and open domain and an element of $\mathcal{A}$.

We aim at determining descent vectorfields $V^* : \mathbb{R}^d \to \mathbb{R}^d$ such that

- $DJ(\Omega)[V^*] < 0$**, and**
- $\tilde{\Omega} = \mathbf{T}_t(\Omega) := (\mathbf{Id} + tV^*)(\Omega)$ **is an open domain.**

Idea: use steepest descent direction V^* for J in the $W^{1,\infty}$ topology[3]:

$$(D_\infty) \qquad V^* = \operatorname*{arg\,min}_{\{V \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d), \|V\|_{1,\infty} \le 1\}} DJ(\Omega)[V].$$

Known practical approaches use Hilbert Space methods with V^* from

$$a(V^*, W) = DJ(\Omega)[W] \text{ for all } W \in H,$$

where $(H, a(\cdot,\cdot))$ denotes an appropriate inner product space[4].

[3] A. Paganini, F. Wechsung, P.E. Farrell. Higher-order moving mesh methods for PDE-constrained shape optimization SISC 40, 2018

[4] G. Allaire, C. Depogny, F. Jouve (2021). Shape and topology optimization. Handbook of Numerical Analysis XXII, Geometric Partial Differential Equations, Part II, p. 1–132

Approaches to solve (D_∞)

A problem of the type of (D) is studied by Ishii and Loreti[5], and also by Capatinelli and Vivaldi[6].

Proposed approaches

- $p-$**Laplace relaxation: consider**

$$(D_p) \qquad \min I_p(v) := \frac{1}{p}\int |DV|^p + DJ(\Omega)[V] \text{ over } V \in W^{1,p}(\mathbb{R}^d,\mathbb{R}^d)$$

  and consider unique solutions V_p of (D_p) as relaxed solutions of (D_∞).

- **Solution formula for the exact solution of (D_∞) in the case $d=1$ [7].**

- **Use ADMM for the numerical solution of (D_∞)[8].**

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[5] H. Ishii and P. Loreti (2005). Limits of solutions of p-Laplace equations as p goes to infinity and related variational problems. Siam J. Math. Anal. 37:411-437.

[6] R. Capitanelli and M.A. Vivaldi (2018). Limit of p-laplacian obstacle problems. arXiv:1811.03863

[7] K. Deckelnick, P.J. Herbert & M. Hinze. A novel $W^{1,\infty}-$approach to shape optimisation with Lipschitz domains. ESAIM: COCV 28 (2) (2022).

[8] e.g. Bartels, S., & Milicevic, M. (2017). Alternating direction method of multipliers with variable step sizes. arXiv preprint arXiv:1704.06069.

Solution framework with hold-all domain[10].

Let us consider the shape optimisation problem

$$\min_{\Omega \in \mathcal{S}} \mathcal{J}(\Omega) = \int_\Omega j(x, u, \nabla u)\, dx \text{ s.t. } \int_\Omega \nabla u \cdot \nabla \eta \, dx = \langle f, \eta \rangle \qquad \text{for all } \eta \in H_0^1(\Omega),$$

where $\mathcal{S} := \{\Omega \subset D \,|\, \Omega = \Phi(\hat{\Omega})$ **for some** $\Phi \in \mathcal{U}\}$**, and**

$$\mathcal{U} := \{\Phi : \bar{D} \to \bar{D} \,|\, \Phi \text{ is a bilipschitz map}, \Phi = \text{id on } \partial D\},$$

Here, $D \subset \mathbb{R}^d$ **open, convex, polygonal hold-all domain, and** $\hat{\Omega} \Subset D$ **a fixed reference domain.**

- **Steepest descent method with Armijo step size rule;**
- **Descent directions from**

$$(D_p) \qquad \underset{\{V \in W^{1,p}(\mathbb{R}^d, \mathbb{R}^d)\}}{\arg\min} \frac{1}{p} \int |DV|^p + DJ(\Omega)[V];$$

and/or

$$(D_\infty) \qquad V^* = \underset{\{V \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d), \|V\|_{1,\infty} \le 1\}}{\arg\min} DJ(\Omega)[V]$$

- **Solution of** (D_∞) **with the alternating direction method of multipliers (ADMM)[9];**
- **Discretization with finite elements;**

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[9] e.g. Bartels, S., & Milicevic, M. (2017). Alternating direction method of multipliers with variable step sizes. arXiv preprint arXiv:1704.06069.

[10] K. Deckelnick, P.J. Herbert & M. Hinze. PDE constrained shape optimisation with first-order and Newton-type methods in the $W^{1,\infty}$ – topology. arXiv:2301.08690 (2023).

Finite element framework and convergence[12].

For the numerical method choose an admissible triangulation $\hat{\mathcal{T}}_h$ of $\bar{D}$ and define

$$\hat{U}_h := \{\Phi_h \in C^0(\bar{D}, \mathbb{R}^d) \,|\, \Phi_{h|\hat{T}} \in P^1(\hat{T}, \mathbb{R}^d), \hat{T} \in \hat{\mathcal{T}}_h, \Phi_h \text{ **is injective**}, \Phi_h = \textbf{id on } \partial D\},$$

and

$$\mathcal{S}_h := \{\Omega_h \subset D; \Omega_h = \Phi_h(\hat{\Omega}) \textbf{ for some } \Phi_h \in \hat{U}_h\},$$

where $\mathcal{T}_{\Omega_h} = \{\Phi_h(\hat{T}),\, \hat{T} \in \hat{\mathcal{T}}_h^{\textbf{ref}}\}$**.**

The discrete shape optimisation problem reads

$$\min_{\Omega_h \in \mathcal{S}_h} \mathcal{J}_h(\Omega_h) = \int_{\Omega_h} j(x, u_h, \nabla u_h)\, dx \textbf{ s.t. } \int_{\Omega_h} \nabla u_h \cdot \nabla \eta_h \, dx = \langle f, \eta_h \rangle \qquad \textbf{for all } \eta_h \in X_{\Omega_h},$$

where

$$X_{\Omega_h} := \{\eta_h \in C^0(\overline{\Omega_h}) \,|\, \eta_{h|T} \in P_1(T), T \in \mathcal{T}_{\Omega_h},\, \eta_h = 0 \textbf{ on } \partial\Omega_h.\}$$

Furthermore, let

$$\mathcal{V}_{\Phi_h} := \{V_h \in C^0(\bar{D}, \mathbb{R}^d) \,|\, V_{h|T} \in P_1(T, \mathbb{R}^d), T = \Phi_h(\hat{T}), \hat{T} \in \hat{\mathcal{T}}_h,\, V_h = 0 \textbf{ on } \partial D\}.$$

Updates then are constructed according to[11]

$$V_h = \arg\min\{\mathcal{J}_h'(\Omega_h)[W_h] \,|\, W_h \in \mathcal{V}_{\Phi_h}, |DW_h| \le 1 \textbf{ in } \bar{D}\},$$

$$\Phi_h^{new} := (\textbf{id} + tV_h) \circ \Phi_h,\; \Omega_h^{new} := (\textbf{id} + tV_h)(\Omega_h).$$

[11] compare also S. Bartels, G. Wachsmuth. Numerical approximation of optimal convex shapes. SISC 42, 2020, and S. Schmidt, V. Schulz. A linear view on shape optimization. SICON 61, 2023.

[12] K. Deckelnick, P.J. Herbert & M. Hinze. Convergence of a steepest descent algorithm in shape optimisation using $W^{1,\infty}$– functions. arXiv:2310.15078 (2023).

Convergence results for the steepest descent method

Theorem 3.3[13]: global convergence of the steepest descent method can be shown for a fixed discretisation parameter, and under mild assumptions also, that every accumulation point of this sequence is a stationary point of the discrete shape functional, i.e.

Let $(\Phi_h^k)_{k\in\mathbb{N}_0} \subset \hat{U}_h$ and $(\Omega_h^k = \Phi_h^k(\hat{\Omega}))_{k\in\mathbb{N}_0} \subset \mathcal{S}_h$ be the sequences generated by the steepest descent method. Then:

(i) $\|\mathcal{J}_h'(\Omega_h^k)\| \to 0$ as $k \to \infty$.

(ii) If $\sup_{k\in\mathbb{N}_0} |(D\Phi_h^k)^{-1}| \le C$, then there exists a subsequence $(\Phi_h^{k_l})_{l\in\mathbb{N}}$, which converges in $W^{1,\infty}(D)$ to a mapping $\Phi_h \in \hat{U}_h$ and $\Omega_h := \Phi_h(\hat{\Omega})$ is a stationary point of $\mathcal{J}_h$, i.e. satisfies $\mathcal{J}_h'(\Omega_h)[V_h] = 0$ for all $V_h \in \mathcal{V}_h$.

Idea of proof: With the Armijo condition at hand, along the lines of Section 2.2.1 of H., Ulbrich, Ulbrich, Pinnau (Optimization with PDE constraints).

[13] K. Deckelnick, P.J. Herbert & M. Hinze. Convergence of a steepest descent algorithm in shape optimisation using $W^{1,\infty}$ – functions. arXiv:2310.15078 (2023).

Convergence results for the steepest descent method cont'd

Theorem 4.4[14]: under suitable conditions a sequence of discrete stationary shapes converges with respect to the Hausdorff complementary metric to a stationary point of the limit problem, i.e.

Suppose that $(\Omega_h)_{0<h\le h_0}$ **satisfies**
(A1) $\forall 0 < h \le h_0 \ \forall V_h \in \mathcal{V}_{\Phi_h}: \quad \mathcal{J}_h'(\Omega_h)[V_h] = 0$**;**

(A2) $\exists M > 1 \ \forall 0 < h \le h_0 \ \forall x, y \in D: \quad M^{-1}|x-y| \le |\Phi_h(x) - \Phi_h(y)| \le M|x-y|$**. Then there exists a sequence** $(h_k)_{k\in\mathbb{N}}$ **with** $\lim_{k\to\infty} h_k = 0$ **and an open set** $\Omega \Subset D$ **such that** $\rho_H^c(\Omega_{h_k}, \Omega) \to 0$ **as** $k \to \infty$**. Furthermore,** Ω **is a stationary point for** $\mathcal{J}$ **on** $\mathcal{S}$**. Here,**

$$\rho_H^c(\Omega_1, \Omega_2) := \max_{x\in\bar{D}} |d_{\complement\Omega_1}(x) - d_{\complement\Omega_2}(x)|$$

denotes the Hausdorff complementarity distance, where $d_{\complement\Omega}(x) := \inf\{|x-y| : y \in \bar{D} \setminus \Omega\}$ **for all** $x \in D$**.**

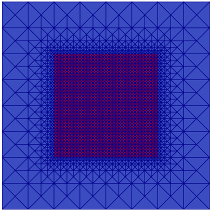
Idea of proof: Continuitiy of the Dirichlet problem w.r.t. the Hausdorff complementarity metric (Mosco convergence), A2 implies uniform convergence (of a subsequence) of the Φ_h **and of the respective domains w.r.t. the HCM. If now the domains are stationary it follows from the structure of the shape derivative and the convergences of the domains, states and co-states that the limit domain is stationary.**

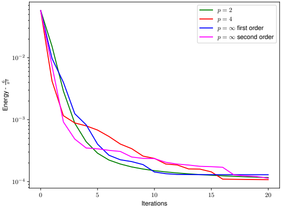
[14] K. Deckelnick, P.J. Herbert & M. Hinze. Convergence of a steepest descent algorithm in shape optimisation using $W^{1,\infty}$ – functions. arXiv:2310.15078 (2023).

Numerical example

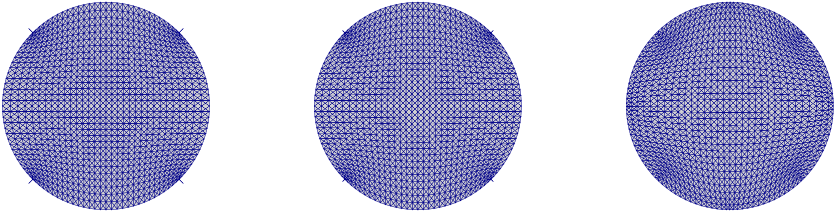
Let $j(x,u,z) := \frac{1}{2}(u-u_d)^2$, where $u_d(x) = \frac{4}{\pi} - |x|^2$ and $f = 1$. Then $-\Delta u_d = 4f$. We expect the minimiser to be given by the ball of radius $\frac{2}{\sqrt{\pi}}$ at the origin which has energy $\frac{6}{\pi^2}$.

Initial domain with cost functional:



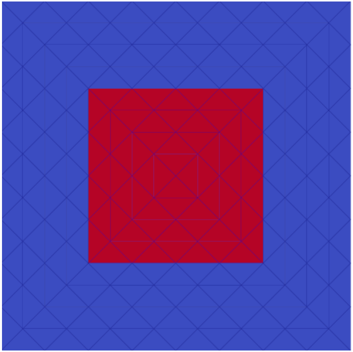


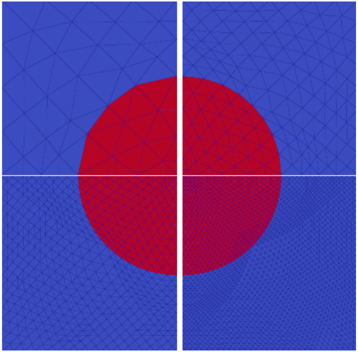
Domains for $p = 2, 4, \infty$:



Experimental order of convergence

Let $j(x,u,z) := \frac{1}{2}|z + \frac{x}{2}|^2$, and $f = 1$. Then $u(x) = \frac{1}{4}(r^2 - |x|^2)$ and $\Omega = B_r(0)$ is optimal with $\mathcal{J}(\Omega) = 0$. We require volume 4 for admissible shapes. The minimiser then is the ball of radius $\frac{2}{\sqrt{\pi}}$ at the origin which has energy 0.





h	μ_h	Energy	EOC Energy	HCD	EOC HCD
0.5	**0.5**	**0.105327**	**–**	**0.0308699**	**–**
0.25	$\frac{1}{\sqrt{2}}$	**0.0268579**	**1.97146**	**0.0273133**	**0.176597**
0.125	**1**	**0.00712922**	**1.91353**	**0.016456**	**0.730993**
0.0625	$\sqrt{2}$	**0.00179752**	**1.98774**	**0.00912092**	**0.851362**
0.03125	**2**	**0.000493593**	**1.86461**	**0.00451768**	**1.0136**

Literature (incomplete selection)

- J. Sokolowski and J.P Zolésio (1992). Introduction to shape optimization. Springer, Berlin

- M.C.Delfour and J.P. Zolésio (2001). Shapes and Geometries – Analysis, Differential Calculus, and Optimization, SIAM

- G. Allaire, C. Depogny, F. Jouve (2021). Shape and topology optimization. Handbook of Numerical Analysis XXII, Geometric Partial Differential Equations, Part II, p. 1–130

- D. Chenais and E. Zuazua. inite-element approximation of 2D elliptic optimal design. Journal de Mathématiques Pures et Appliquées 85.2 (2006)

- S. Bartels and G. Wachsmuth. Numerical Approximation of Optimal Convex Shapes. SIAM Journal on Scientific Computing 42.2 (2020)

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- W. Gong and S. Zhu. On discrete shape gradients of boundary type for PDE-constrained shape optimization. SIAM Journal on Numerical Analysis 59.3 (2021)

- R. Hiptmair, A. Paganini, and S. Sargheini. Comparison of approximate shape gradients. BIT Numerical Mathematics 55.2 (2015)

- B. Kiniger and B. Vexler. A priori error estimates for finite element discretizations of a shape optimization problem. ESAIM: M2AN 47.6 (2013)

- S. Zhu and Z. Gao. Convergence analysis of mixed finite element approximations to shape gradients in the Stokes equation. Computer Methods in Applied Mechanics and Engineering 343 (2019)

- S. Zhu, X. Hu, and Q. Liao. Convergence analysis of Galerkin finite element approximationsto shape gradients in eigenvalue optimization. BIT Numerical Mathematics 60 (2020)

- G. Starke. Shape Optimization by Constrained First-Order Least Mean Approximation. arXiv preprint arXiv:2309.13595 (2023)
- …

Summary

We propose a finite element framework for PDE constrained shape optimization in the $W^{1,\infty}$ topology.

- **We prove global convergence of steepest descent with Armijo step size rule in the discrete setting.**
- **Under mild assumptions we prove convergence in the Hausdorff complementary metric of discrete stationary shapes to a stationary point of the continuous problem.**

Literature:

- **K. Deckelnick, P. Herbert, M. Hinze: A novel $W^{1,\infty}-$approach to shape optimisation with Lipschitz domains. ESAIM: COCV 28 (2) (2022).**
- **P.M. Müller, N. Kühl, M. Siebenborn, K. Deckelnick, M. Hinze, T. Rung: A novel $p-$Harmonic Descent Approach applied to Fluid Dynamic Shape Optimization. Structural and Multidisciplinary Optimization 64 (2021).**
- **K. Deckelnick, P.J. Herbert, M. Hinze: Shape optimisation with first-order and Newton-type methods in the $W^{1,\infty}$ topology arXiv.2301.08690 (2023).**
- **K. Deckelnick, P.J. Herbert, and M. Hinze. Convergence of a steepest descent algorithm in shape optimisation using $W^{1,\infty}$ functions. arXiv.2310.15078 (2023)**

Related: Philip Herbert's talk at 10:00 on Thursday: A combined diffuse interface and sharp interface method for shape optimisaton.

Thank you for your attention.