Free Boundary Problems for Viscous Incompressible Fluids via Da Prato-Grisvard Theory

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Free Boundary Problems for Fluids

- describe evolution of fluid with free surfaces $\Gamma(t)$
- surfaces $\Gamma(t)$ are given only at initial time, their determination for $t > 0$ is part of the problem
- in this talk : viscous, incompressible fluid and compressible pressureless gas
- balance laws on free boundary with/without surface tension : : $T(v,p)\nu = \sigma H \nu$ or $T(v,p)\nu = 0$
- Cauchy stress tensor $T(v,p) = 2\mu D(v) - pI$ or $T = 2\mu(|D(v)|^2)D(v)$
- viscous stress tensor$S(v) = \mu D(v) + \lambda \text{div } vI, \quad \mu > 0, 2\mu + n\lambda > 0$
- Deformation tensor $D(v) = \frac{1}{2}[(\nabla v) + (\nabla v)^T]$
- long history : Solonnikov, Beale, Tani,..., Guo, Tice, ..., Pruss, Simonett,..., Shibata,...., Ogawa, Shimizu, ...

Free Boundary Value Problems for Newtonian Fluids

The situation for Newtonian fluids without surface tension

$$\begin{cases} v_t + (v \cdot \nabla)v - \text{div } T(v,p) &= 0, & t > 0, x \in \Omega_t \\ \text{div } v &= 0, & t > 0, x \in \Omega_t \\ T(v,p)\nu &= 0, & t > 0, x \in \Omega_t \\ v \cdot \nu &= -(\partial_t \eta)/|\nabla_x \eta|, & t > 0, x \in \Omega_t \\ v(0) &= v_0, & x \in \Omega_0 \\ \Omega_t(0) &= \Omega_0, \end{cases}$$

where

- Cauchy stress tensor $T(v,p) = D(v) - pI$
- Deformation tensor $D(v) = \frac{1}{2}[(\nabla v) + (\nabla v)^T]$
- outer unit normal ν
- $\partial\Omega_t = \{x \in \mathbb{R}^n : \eta(t,x) = 0\}$

Lagrangian Approach to Newtonian Fluids

$$\begin{cases} v_t + (v\cdot\nabla)v - \operatorname{div}\, T(v,p) &= 0, & t>0, x\in\Omega_t \\ \operatorname{div}\, v &= 0, & t>0, x\in\Omega_t \\ T(v,p)\nu &= 0, & t>0, x\in\Omega_t \\ v\cdot\nu &= -(\partial_t\eta)/|\nabla_x\eta|, & t>0, x\in\Omega_t \\ v(0) &= v_0, & x\in\Omega_0 \\ \Omega_t(0) &= \Omega_0, & \end{cases}$$

- consider unbounded initial domains as $\Omega_0 = \mathbb{R}^n_+$
- trajectory given by $X_u(t,\xi) = \xi + \int_{t_0}^t u(s,\xi)ds$, u velocity
- well-defined coordinate transform between Eulerian and Lagrangian variables provided $\xi \mapsto X_u(t,\xi)$ is invertible
- yes, if Jacobian $D_\xi X_u(t,\xi) = Id + \int_{t_0}^t D_\xi u(s,\xi)ds$ is invertible
- need $c<1$ such that $\| \int_{t_0}^t D_\xi(u,\xi)ds\|_\infty \le c$
- global transform : need $L_1(\mathbb{R}_+; L_\infty(\Omega))$-bound on Jacobian
- motivates wish for L^1-maximal regularity theory
- if above follows from L_q-estimate $(1<q<\infty)$, then c depends on t

Discussion and Heuristics

- standard $L^q(J;X)$ regularity theory fails for $q=1$ if X reflexive
- Da Prato - Grisvard : Max. L^q-regularity for $q \in [1,\infty)$ for free if A is invertible and X is replaced by $D_A(\theta,q)$
- $D_A(\theta,q) = \{x \in X : [x]_{\theta,q} := (\int_0^\infty |t^{1-\theta} A e^{tA} x|_X^q \frac{dt}{t})^{1/q} < \infty\}$
- $D_A(\theta,q) = (X, D(A))_{\theta,q}$
- in our setting : no exponential decay for e^{tA}
- Aim : develop homogeneous version : global $L_1(\mathbb{R}_+;(X,D(A))_{\theta,q})$-regularity provided $(X,D(A))_{\theta,q}$ carries homogeneous norm $[\cdot]_{\theta,q}$ and A non invertible
- resulting space : homogenous Besov space $\dot{B}^{2\theta}_{p,q}(\mathbb{R}^n_+)$
- only homogeneous Besov space that embedds into L^∞ is $\dot{B}^{n/p}_{p,1}$
- Lagrange : aim for setting such that $\nabla u \in \dot{B}^{n/p}_{p,1}$, hence $\nabla^2 u \in \dot{B}^{n/p-1}_{p,1}$
- summary : seems that $\dot{B}^{n/p-1}_{p,1}$ is good choice of ground space for maximal L^1-regularity

Interpolation of Homogeneous Spaces

- homogeneous version of Da Prato-Grisvard in $\dot{D}_A(\theta, q) = (X, D(\dot{A}))_{\theta,q}$ = abstract homogeneous Besov space
- here : use for X homogeneous Sobolev scale
- $(\dot{H}^0_p(\mathbb{R}^n), \dot{H}^m_p(\mathbb{R}^n))_{\theta,q} = \dot{B}^{m\theta}_{p,q}(\mathbb{R}^n)$
- extension to half spaces
- extension to solenoidal spaces

The linear Stokes equation

- linear Stokes problem : $n \geq 2$, $1 < p < \infty$, $s \in (0,1)$ with $s < n/p$:

$$\begin{cases} v_t - \text{div } T(v,p) & = f, \quad t > 0, x \in \mathbb{R}^n_+ \\ \text{div } v & = 0, \quad t > 0, x \in \mathbb{R}^n_+ \\ T(v,p)e_n & = 0, \quad t > 0, x \in \partial\mathbb{R}^n_+ \\ v(0) & = v_0, \quad x \in \mathbb{R}^n_+ \end{cases}$$

admits maximal $L_1(\mathbb{R}_+; \dot{B}^s_{p,1}(\mathbb{R}^n_+))$-regularity

- this means : for all $f \in L^1(\mathbb{R}_+; \dot{B}^s_{p,1,\sigma})$ and all $u_0 \in \dot{B}^s_{p,1,\sigma}(\mathbb{R}^n_+))$ there exists a unique solution u to Stokes equation satisfying

$$\|u\|_{C_b(\mathbb{R}_+;\dot{B}^s_{p1\sigma}(\mathbb{R}^n_+))} + \|u_t, \nabla^2 u\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p1}(\mathbb{R}^n_+))} + \|\nabla p\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p1}(\mathbb{R}^n_+))}$$
$$\leq C[\|f\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p1}(\mathbb{R}^n_+))} + \|u_0\|_{\dot{B}^s_{p1}(\mathbb{R}^n_+))}]$$

- related results by Ogawa-Shimizu (2024) by different approach : explicit solution formula and adapted Littlewood-Paley theory
- note : $L^q - L^p$ estimates or $L^2 - L^2$-energy estimates impossible for $[0, \infty)$

Back to Free Boundary Problem

Theorem(Memoirs AMS)

Let $v_0 \in \dot{B}^{n/p-1}_{p,1}(\mathbb{R}^n_+)$ for $p \in (n-1, n)$. Then there exists $c > 0$ such that if

$$\|v_0\|_{\dot{B}^{n/p-1}_{p,1}(\mathbb{R}^n_+)} \leq c,$$

then FBP for viscous, incompressible Newtonian fluids

$$\begin{cases} v_t + (v \cdot \nabla)v - \text{div } T(v,p) &= 0, & t > 0, x \in \Omega_t \\ \text{div } v &= 0, & t > 0, x \in \Omega_t \\ T(v,p)\nu &= 0, & t > 0, x \in \Omega_t \\ v \cdot \nu &= -(\partial_t \eta)/|\nabla_x \eta|, & t > 0, x \in \Omega_t \\ v(0) &= 0, & x \in \Omega_0 \\ \Omega_t(0) &= \mathbb{R}^n_+, \end{cases}$$

admits a unique, global solution (v, p, Ω_t) in the maximal regularity class $L_1(\mathbb{R}_+; \dot{B}^{n/p-1}_{p,1}(\Omega_t))$ with $\Omega_t = X_v(t, \mathbb{R}^n_+)$

Remark 1 : $\dot{B}^{n/p-1}_{p,1}(\mathbb{R}^n_+)$ is scaling invariant critical space

Remark 2 : IVP : Fujita-Kato $\dot{H}^s_p$, Giga, Kato L^n, Cannone $\dot{B}^{n/p-1}_{pp}$

Remark 3 : approach based on homogeneous version of Da Prato-Grisvard

Remark 4 : difficulty : estimate nonlinear terms in $\dot{B}^s_{p,1}(\mathbb{R}^n_+)$

Idea of Proof

- FBP in Lagragian coordinates reads as

$$\begin{cases} u_t - \text{div}\,_u \mathbb{T}_u(u,P) &= 0, \quad t>0, x\in\mathbb{R}^n_+ \\ \text{div}\,_u u &= 0, \quad t>0, x\in\mathbb{R}^n_+ \\ \mathbb{T}(u,P)n_u &= 0, \quad t>0, x\in\partial\mathbb{R}^n_+ \\ u(0) &= u_0, \quad x\in\Omega_0 \end{cases}$$

where

- $A_u(t,y) = (D_y X_u(t,y))^{-1}$
- $\nabla_u = A_u^T \nabla y$, $\text{div}\,_u = A_u^T : \nabla_y$
- $\mathbb{T}_u(w,\theta) = \nabla_u w + (\nabla_u w)^T - \theta Id$
- $n_u(t,y) = n(t, X_u(t,y))$

Idea of Proof II

Rewrite Lagragian version by putting all linear terms to left side

$$
\begin{cases}
u_t - \mathrm{div}\ _u\mathbb{T}_u(u,P) &= f, \quad t > 0, x \in \mathbb{R}^n_+ \\
\mathrm{div}\ _u u &= g, \quad t > 0, x \in \mathbb{R}^n_+ \\
\mathbb{T}(u,P)e_n|_{\partial\mathbb{R}^n} &= h, \quad t > 0, x \in \partial\mathbb{R}^n_+ \\
u(0) &= u_0, \quad x \in \Omega_0
\end{cases}
$$

where

- $f = \mathrm{div}\ _u\mathbb{T}_u(u,P) - \mathrm{div}\ T(u,P)$
- $g = \mathrm{div}\ u - \mathrm{div}\ _u u$
- $h = T(u,P)e_n - \mathbb{T}_u(u,P)n_u$

and apply homogeneous Da Prato-Grisvard theorem with contraction principle

FBP for Pressureless Gas

$$\begin{cases} \varrho(v_t + (v\cdot\nabla)v) - \text{div } S(v) &= 0, & t>0, x\in\Omega_t \\ \varrho_t + \text{div } (\varrho v) &= 0, & t>0, x\in\Omega_t \\ S(v)\nu &= 0, & t>0, x\in\Omega_t \\ v\cdot\nu &= -(\partial_t\eta)/|\nabla_x\eta|, & t>0, x\in\Omega_t \\ v(0) &= v_0, & x\in\Omega_0 \\ \Omega_t(0) &= \Omega_0, \end{cases}$$

where

- viscous stress tensor $S(v) = \mu D(v) + \lambda \text{div } vI$, $\quad \mu > 0, 2\mu + n\lambda > 0$
- proceed as before : show maximal L^1-regularity for Lamé-system

$$\begin{cases} u_t - \text{div } S(u) &= f, & t>0, x\in\mathbb{R}^n_+ \\ S(u)e_n|_{\partial\mathbb{R}^n_+} &= g, & t>0, x\in\partial\mathbb{R}^n_+ \\ u(0) &= u_0, & x\in\mathbb{R}^n_+ \end{cases}$$

L^1-regularity for Lamé-system

- linear Lamé system : $n \geq 2$, $1 < p < \infty$, $s \in (0, 1/p)$ with $s < n/p - 1$:
  Lamé system admits maximal $L_1(\mathbb{R}_+; \dot{B}^s_{p,1}(\mathbb{R}^n_+))$-regularity
- this means : for all $f \in L^1(\mathbb{R}_+; \dot{B}^s_{p,1,\sigma})$, $g \in Y := \dot{B}^{1/2(s+1-1/p)}_{11}(\mathbb{R}_+; \dot{B}^0_{p,1}(\partial\mathbb{R}^n_+)) \cap L^1(\mathbb{R}_+; \dot{B}^{s+1=1/p}_{p,1})$ and all $v_0 \in \dot{B}^s_{p,1,\sigma}(\mathbb{R}^n_+))$ there exists a unique solution v to Lamé satisfying

$$\|u\|_{C_b(\mathbb{R}_+;\dot{B}^s_{p1\sigma}(\mathbb{R}^n_+))} + \|u_t, \nabla^2 u\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p1}(\mathbb{R}^n_+))} + \leq C[\|f\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p1}(\mathbb{R}^n_+))} + \|g\|_Y + \|v_0\|_{\dot{B}^s_{p1}(\mathbb{R}^n_+))}]$$

Back to FBP : Global solution for small data

Theorem : (Memoirs AMS)

FBP for pressureless gas admits unique, global solution in $L^1(\mathbb{R}_+; \dot{B}^{n/p-1}_{p,1})$ provided data are small enough in $\dot{B}^{n/p-1}_{p,1}(\mathbb{R}^n_+)$

Remark : approach based again on homogeneous version of Da Prato-Grisvard theorem