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Mathematical Aspects for Interfaces and Free Boundaries
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Well-posedness of Hele-Shaw type moving boundary problem associated with gradient method for shape optimization

– an example of well-posed shape optimization flow –

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Shape optimization via the cost functional $J(\Omega)$

Shape derivative (shape gradient) G

$$
\begin{aligned}
dJ(\Omega)[\mathbf{V}] &= \lim_{t \searrow 0} \frac{J(\Omega_t) - J(\Omega)}{t} \\
&= \frac{d}{dt} J(\Omega_t) \Big|_{t=0} \\
&= \int_\Gamma G\nu \cdot \mathbf{V} \, ds,
\end{aligned}
$$

Shape optimization algorithm ($\Delta t > 0$)

$$
\Gamma^{k+1} := \{ x - \Delta t (G\nu)|_{\Gamma^k}(x) \mid x \in \Gamma^k \}
$$

$$
\wr\wr
$$

$$
V = -G\nu : \text{Shape optimization flow}
$$

$$
\min_{\Omega} J(\Omega)
$$

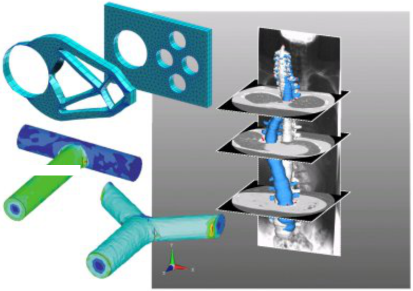
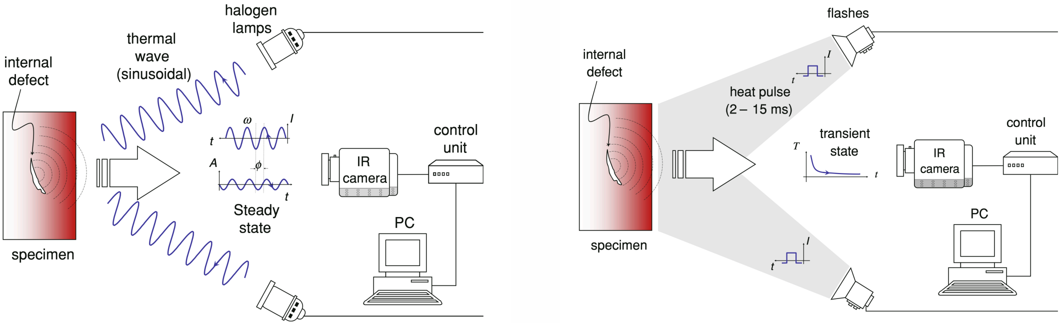


Figure: Examples of optimal shape design in engineering from Prof.Hideyuki Azegami's web site: http://www.az.cs.is.nagoya-u.ac.jp/research.html

Non-Destructive Testing (NDT) and Evaluation: Thermal Imaging[1]



In thermal imaging and NDT purposes,

- **lock-in thermography** (left figure) relies on modulated heating and synchronous detection to detect surface defects with high sensitivity.
- **Pulse thermography** (right figure) utilizes a short heat pulse and analyzes the material's cooling behavior to detect subsurface defects.

[1] Photos taken from [Clemente Ibarra-Castanedo et al 2013 Eur. J. Phys. 34 S91]

Physical Model

Let $D \in \mathbb{R}^d$ be a simply connected domain with boundary $\Sigma = \partial D$ and assume that an unknown simply connected inclusion ω with regular boundary $\Gamma = \partial\omega$ is located inside the domain D satisfying $\mathrm{dist}(\Sigma, \Gamma) > 0$, see figure.

To determine the inclusion ω, we measure for a given current distribution $g \in H^{-1/2}(\Sigma)$ the voltage distribution $f \in H^{1/2}(\Sigma)$ at the boundary Σ. Hence, we are seeking a domain $\Omega := D \backslash \overline{\omega}$ and an associated harmonic function u, satisfying the overdetermined boundary value problem

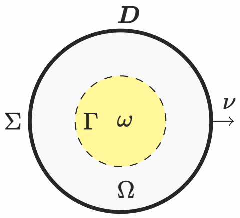


Figure: Conceptual model

$$\text{(IP)} \qquad \begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \\ u = f \quad \text{and} \quad \nabla u \cdot \nu = g & \text{on } \Sigma. \end{cases}$$

Theorem 1 (Identifiability result, [BD10])

The Cauchy pair $(f, g) \neq (0, 0)$ *uniquely determine* Γ *and* u *satisfying* (IP).

Kohn-Vogelius method [RS96]

$$J(\Omega) = \int_{\Omega} |\nabla(u_{\mathrm{D}} - u_{\mathrm{N}})|^2 \, dx \ \to \inf$$

where u_{D} and u_{N} respectively solves

(D)
$$\begin{cases} -\Delta u_{\mathrm{D}} = 0 & \text{in } \Omega, \\ u_{\mathrm{D}} = 0 & \text{on } \Gamma, \\ u_{\mathrm{D}} = f & \text{on } \Sigma. \end{cases}$$

(N)
$$\begin{cases} -\Delta u_{\mathrm{N}} = 0 & \text{in } \Omega, \\ u_{\mathrm{N}} = 0 & \text{on } \Gamma, \\ \nabla u_{\mathrm{N}} \cdot \nu = g & \text{on } \Sigma. \end{cases}$$

Other choices of the cost functional

$$J_1(\Omega) := \int_{\Sigma} \left| \frac{\partial u_D}{\partial \nu} - g \right|^2 ds$$

$$J_2(\Omega) := \int_{\Sigma} |u_N - f|^2 \, ds$$

Shape derivative of J [RS96]

Let the underlying variation fields $\mathbf{V}$ be sufficiently smooth such that a $C^{1,1}$-regularity is preserved for all the perturbed domains. Then,

$$\begin{aligned} dJ(\Omega)[\mathbf{V}] &= \lim_{t \searrow 0} \frac{J(\Omega_t) - J(\Omega)}{t} \\ &= \frac{d}{dt} J(\Omega_t) \Big|_{t=0} \\ &= \int_{\Gamma} G\nu \cdot \mathbf{V} \, ds, \end{aligned}$$

where

$$G := G^+ G^- = \left(\frac{\partial u_{\mathrm{D}}}{\partial \nu} + \frac{\partial u_{\mathrm{N}}}{\partial \nu} \right) \left(\frac{\partial u_{\mathrm{D}}}{\partial \nu} - \frac{\partial u_{\mathrm{N}}}{\partial \nu} \right).$$

Shape optimization approach

Kohn-Vogelius method [RS96]

$$J(\Omega) = \int_{\Omega} |\nabla(u_{\mathrm{D}} - u_{\mathrm{N}})|^2 \, dx \ \to \inf$$

where u_{D} and u_{N} respectively solves

$$\text{(D)} \qquad \begin{cases} -\Delta u_{\mathrm{D}} = 0 & \text{in } \Omega, \\ u_{\mathrm{D}} = 0 & \text{on } \Gamma, \\ u_{\mathrm{D}} = f & \text{on } \Sigma. \end{cases}$$

$$\text{(N)} \qquad \begin{cases} -\Delta u_{\mathrm{N}} = 0 & \text{in } \Omega, \\ u_{\mathrm{N}} = 0 & \text{on } \Gamma, \\ \nabla u_{\mathrm{N}} \cdot \nu = g & \text{on } \Sigma. \end{cases}$$

Shape derivative of J [RS96]

Let the underlying variation fields $\mathbf{V}$ be sufficiently smooth such that a $C^{1,1}$-regularity is preserved for all the perturbed domains. Then,

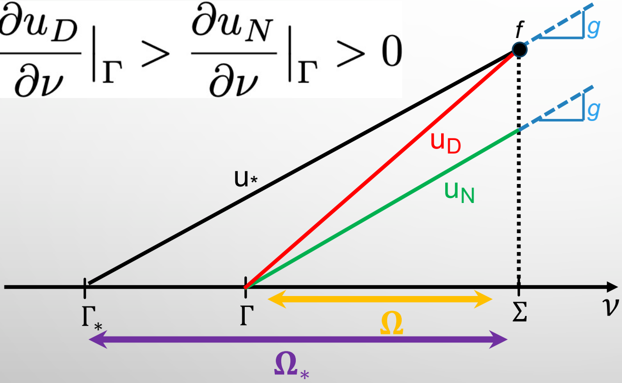
$$\begin{aligned} dJ(\Omega)[\mathbf{V}] &= \lim_{t \searrow 0} \frac{J(\Omega_t) - J(\Omega)}{t} \\ &= \frac{d}{dt} J(\Omega_t) \Big|_{t=0} \\ &= \int_{\Gamma} G\nu \cdot \mathbf{V} \, ds, \end{aligned}$$

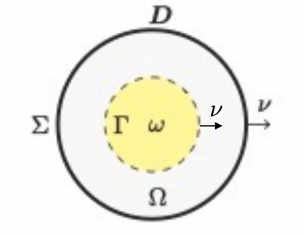
where

$$G := G^+ G^- = \left(\frac{\partial u_{\mathrm{D}}}{\partial \nu} + \frac{\partial u_{\mathrm{N}}}{\partial \nu} \right) \left(\frac{\partial u_{\mathrm{D}}}{\partial \nu} - \frac{\partial u_{\mathrm{N}}}{\partial \nu} \right).$$

If $^{\exists}(\Omega_*, u_*)$: exact solution, $\partial\Omega_* = \Sigma \cup \Gamma_*$ and $\Gamma \subset \Omega_*$, then

$$\frac{\partial u_D}{\partial \nu}\Big|_\Gamma > \frac{\partial u_N}{\partial \nu}\Big|_\Gamma > 0$$





$$\begin{cases} -\Delta u_D = 0 & \text{in } \Omega, \\ u_D = 0 & \text{on } \Gamma, \\ u_D = f & \text{on } \Sigma. \end{cases}$$

$$\begin{cases} -\Delta u_N = 0 & \text{in } \Omega, \\ u_N = 0 & \text{on } \Gamma, \\ \nabla u_N \cdot \nu = g & \text{on } \Sigma. \end{cases}$$

Choice of descent vector [EH05]

Suppose $\mathbf{0} \not\equiv \mathbf{V} = -G\nu \in L^2(\Gamma)^d$. Then, formally, for sufficiently small $t > 0$ we have

$$\begin{aligned} J(\Omega_t) &= J(\Omega) + t dJ(\Omega)[\mathbf{V}] + O(t^2) \\ &= J(\Omega) + t \int_\Gamma G\nu \cdot \mathbf{V} \, ds + O(t^2) \\ &= J(\Omega) - t \int_\Gamma |\mathbf{V}|^2 \, ds + O(t^2) \\ &< J(\Omega). \end{aligned}$$

The choice $\mathbf{V} = -G\nu$ as the descent vector is straightforward and practical.

Different choice of descent vector [SKR22]

If $f > 0$ on Σ, then
and g > 0

$$G^+ = \frac{\partial u_\mathsf{N}}{\partial \nu} + \frac{\partial u_\mathsf{D}}{\partial \nu} > 0 \quad \text{on } \Gamma.$$

Choosing

$$\mathbf{V} = -G^-\nu = -\left(\frac{\partial u_\mathsf{D}}{\partial \nu} - \frac{\partial u_\mathsf{N}}{\partial \nu} \right) \nu$$

we see that

$$\begin{aligned} J(\Omega_t) &= J(\Omega) + t \int_\Gamma G^+ G^- \nu \cdot \mathbf{V} \, ds + O(t^2) \\ &= J(\Omega) + t \int_\Gamma \underbrace{G^+}_{>0} |\mathbf{V}|^2 \, ds + O(t^2) \\ &< J(\Omega). \end{aligned}$$

Algorithm

1. Initialization: Fix a small $\Delta t > 0$ and choose an initial shape Γ_0.
2. Iteration: For $k = 0, 1, 2, \ldots$:
   - 2.1 Solve (D) and (N) on Ω_k.
   - 2.2 Set $V_{n,k} := -\left(\nabla u_{\mathrm{D},k} \cdot \nu_k - \nabla u_{\mathrm{N},k} \cdot \nu_k\right)$ on Γ^k.
   - 2.3 Set $\Gamma_{k+1} = \{x + \Delta t \mathbf{V}_k(x) \mid x \in \Gamma_k\}$.
3. Stop Test: Repeat *Iteration* until convergence.

Comoving Mesh Method (CMM) [SKR22, SRK24]
(FE scheme for general MBP in 2d/3d)

Let $T > 0$, $N_T > 0$ be an integer, and $\Delta t := T/N_T$.

For each $k = 0, 1, \cdots, N_T$, let

- $t_k = k\Delta t$,
- $\Omega_k \approx \Omega(k\Delta t)$,
- $\Gamma_k \approx \Gamma(k\Delta t)$,
- $u_{\mathrm{D},k} \approx u_{\mathrm{D}}(\cdot, k\Delta t)$,
- $u_{\mathrm{N},k} \approx u_{\mathrm{N}}(\cdot, k\Delta t)$.

Then, given Γ_0, the previous algorithm reduces to

$$
\begin{cases}
-\Delta u_{\mathrm{D},k} = 0, & \text{in } \Omega_k \\
u_{\mathrm{D},k} = f, & \text{on } \Sigma, \\
u_{\mathrm{D},k} = 0, & \text{on } \Gamma_k \\
-\Delta u_{\mathrm{N},k} = 0, & \text{in } \Omega_k \\
\nabla u_{\mathrm{N},k} \cdot \nu_k = g, & \text{on } \Sigma, \\
u_{\mathrm{N},k} = 0, & \text{on } \Gamma_k \\
V_{n,k} = -\left(\nabla u_{\mathrm{D},k} \cdot \nu_k - \nabla u_{\mathrm{N},k} \cdot \nu_k\right) & \text{on } \Gamma_k \\
\Gamma(0) = \Gamma_0, &
\end{cases}
$$

1. Initialization: Fix a small $\Delta t > 0$ and choose an initial shape Ω_0.
2. Iteration: For $k = 0, 1, 2, \ldots$:
   - 2.1 Solve (D) and (N) on Ω_k.
   - 2.2 Compute $\mathbf{V}_k \in V(\Omega_k)^d$ by solving the variational equation

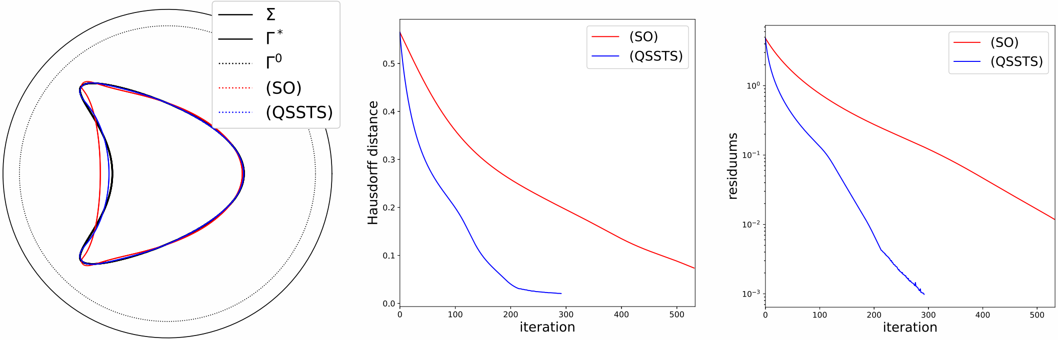
$$a(\mathbf{V}_k, \varphi) = \int_{\Gamma_k} \tilde{G}_k \nu_k \cdot \varphi \, ds, \quad \forall \varphi \in V(\Omega_k)^d,$$

     where $V(\Omega_k) := \{\varphi \in H^1(\Omega_k) \mid \varphi = 0 \text{ on } \Sigma\}$ and a is a bounded and coercive bilinear form on $V(\Omega_k)^d$.
   - 2.3 Set $\Omega_{k+1} = \{x + \Delta t \mathbf{V}_k(x) \mid x \in \Omega_k\}$.
3. Stop Test: Repeat *Iteration* until convergence.

Remark 1

In step 2.2, we can choose either $\tilde{G} = G$ or, if $f > 0$, $\tilde{G} = G^-$.

A numerical experiment: classical versus proposed method



Note Here, we used a non-uniform time step size to clearly highlight the potential of taking $\tilde{G} = G^{-}$. In fact, we calculate the step size at each time step using a backtracking line search:

$$\Delta t_k = c \frac{J(\Omega_k)}{|\mathbf{V}_k|^2_{\mathbf{H}^1(\Omega_k)}},$$

where $c > 0$ is a scaling factor.

For fix $T > 0$, and given $f \geq 0$ ($f \neq 0$), $g \geq 0 (g \neq 0)$, Σ, and Γ_0, we observe that, in the **continuous setting**, the boundary variation algorithm yields the following moving boundary problem:

(HSP)
$$\begin{cases} -\Delta u_{\mathrm{D}}(x,t) = 0, & x \in \Omega(t), \quad t \in [0,T], \\ u_{\mathrm{D}}(x,t) = f(x), & x \in \Sigma, \\ u_{\mathrm{D}}(x,t) = 0, & x \in \Gamma(t), \quad t \in [0,T], \\ -\Delta u_{\mathrm{N}}(x,t) = 0, & x \in \Omega(t), \quad t \in [0,T], \\ \dfrac{\partial}{\partial \nu} u_{\mathrm{N}}(x,t) = g(x), & x \in \Sigma, \\ u_{\mathrm{N}}(x,t) = 0, & x \in \Gamma(t), \quad t \in [0,T], \\ V_n(x,t) = -\left[\dfrac{\partial}{\partial \nu} u_{\mathrm{D}}(x,t) - \dfrac{\partial}{\partial \nu} u_{\mathrm{N}}(x,t) \right] & x \in \Gamma(t), \quad t \in [0,T], \\ \Gamma(0) = \Gamma_0, \end{cases}$$

where $V_n(x,t)$ represents the velocity of movement of $\Gamma(t)$ in the direction of the normal $\nu(t)$ to $\Gamma(t)$, for all $t > 0$.

- Shape inverse problems are typically solved numerically through shape optimization; see [RS96, EH05].
- The Hele-Shaw-like system (HSP) is a specific case of the general conductivity reconstruction problem which is severely ill-posed in the sense of Hadamard [EH05].
- Despite this, it has been widely studied both theoretically and numerically; see [EH05, Afr22, AK02, AIP95, AV96, BD10, CK05, HR98, Isa66].
- The existence and uniqueness of the solution from boundary measurement data have been examined by several authors; see [AIP95, AV96, BD10, Isa66].
- Shape optimization reformulations of shape inverse problems are rarely examined from different theoretical and numerical perspectives.
- This investigation aims to rigorously analyze the **existence, uniqueness, and continuous dependence on the data** of the classical solution of (HSP) in a short-time horizon.
- Little to no work has been done on the **well-posedness** of the shape optimization problem from which (HSP) is derived, especially in the direction of our study.
- The **system** (HSP), derived from a shape optimization context and originating from a shape inverse problem, **is novel.**
- **Our analysis**, inspired by Bizhanova and Solonnikov [BS00] and Solonnikov [Sol03], **offers a new perspective.**

- $D \subset \mathbb{R}^d$ be a bounded (simply connected) domain D with boundary $\partial D = \Sigma$
- $\mathcal{A}^{2+\alpha} := \{\Gamma = \partial\omega \mid \overline{\omega} \subset D,\ \omega$ is a simply connected bounded domain and $\partial\omega \in C^{2+\alpha}\}$.
- For $\Gamma \in \mathcal{A}^{2+\alpha}$, $\Omega(\Gamma)$ denotes an annular domain in $\mathbb{R}^d$ with boundary $\partial\Omega(\Gamma) = \Gamma \cup \Sigma$.

Given $f \in C^{2+\alpha}(\Sigma)$ and $g \in C^{1+\alpha}(\Sigma)$, $u_{\mathrm{D}}(\Gamma)$ and respectively solves

(D)
$$\begin{cases} u_{\mathrm{D}} \in C^{2+\alpha}(\overline{\Omega(\Gamma)}) & \\ -\Delta u_{\mathrm{D}} = 0, & \text{in } \Omega(\Gamma) \\ u_{\mathrm{D}} = f, & \text{on } \Sigma, \\ u_{\mathrm{D}} = 0, & \text{on } \Gamma. \end{cases}$$

(N)
$$\begin{cases} u_{\mathrm{N}} \in C^{2+\alpha}(\overline{\Omega(\Gamma)}) & \\ -\Delta u_{\mathrm{N}} = 0, & \text{in } \Omega(\Gamma) \\ \nabla u_{\mathrm{N}} \cdot \nu = g, & \text{on } \Sigma, \\ u_{\mathrm{N}} = 0, & \text{on } \Gamma. \end{cases}$$

Hereafter, unless otherwise stated, we assume $\Gamma \in \mathcal{A}^{2+\alpha}$.

Preliminaries: quasi-normal vectors on Γ and a diffeomorphic map

Definition 2

We say that a vector field $\mathbf{N}$ is quasi-normal on $\Gamma \in \mathcal{A}^{2+\alpha}$, inheriting the regularity of Γ, if

(1) $\mathbf{N} \in C^{2+\alpha}(\Gamma; \mathbb{R}^d)$ and it is such that $|\mathbf{N}(\xi)| = 1$ and $\mathbf{N}(\xi) \cdot \nu(\xi; \Gamma) > 0$ for all $\xi \in \Gamma$.

We let $I_{\varepsilon_0} := [-\varepsilon_0, \varepsilon_0]$ and fix a constant $\varepsilon_0 = \varepsilon_0(\Gamma, \mathbf{N}) > 0$ such that the map

$$X : \Gamma \times I_{\varepsilon_0} \longrightarrow \Gamma^{\varepsilon_0} \subset \mathbb{R}^d, \qquad X(\xi, \rho) \longmapsto \xi + \rho \mathbf{N}(\xi) \subset D,$$

is a $C^{2+\alpha}$-diffeomorphism, where

$$\Gamma^{\varepsilon} := \{X(\xi, r) := \xi + r\mathbf{N}(\xi) \mid (\xi, r) \in \Gamma \times I_{\varepsilon}\},$$

for $\varepsilon > 0$.

Proposition 1

There exists a constant $\varepsilon_0 > 0$ such that

$$X \in \mathrm{Diffeo}^{2+\alpha}(\Gamma \times \overline{I}_{\varepsilon_0}; D), \qquad D := X(\Gamma \times \overline{I}_{\varepsilon_0}).$$

For fixed real numbers a, b where $b > a$, the scalar valued ρ is such that it belongs to the Banach space

$$R_{[a,b]}(\Gamma, \mathbf{N}) := \left\{\rho : \Gamma \times [a,b] \to I_{\varepsilon_0(\Gamma,\mathbf{N})} \mid \rho \in C([a,b]; C^{2+\alpha}(\Gamma)) \cap C^1([a,b]; C^{1+\alpha}(\Gamma))\right\}.$$

We also introduce the set

$$R_0(\Gamma, \mathbf{N}) := \left\{\rho \in C^{2+\alpha}(\Gamma) \mid |\rho(\xi)| \leqslant \varepsilon_0(\Gamma, \mathbf{N}),\ \forall \xi \in \Gamma\right\}.$$

For $\rho \in R(\Gamma, \mathbf{N})$, it can be shown that $\mathcal{S}(\rho) := \{X(\xi, \rho) \mid \xi \in \Gamma\}$ **is a $C^{2+\alpha}$ boundary.**

Proposition 2

There exists $\varepsilon_1 := \varepsilon_1(\Gamma, \mathbf{N}) \in (0, \varepsilon_0(\Gamma, \mathbf{N})]$ such that $\mathcal{S}(\rho) \in \mathcal{A}^{2+\alpha}$ holds for $\rho \in R_1(\Gamma, \mathbf{N})$, where

$$R_1(\Gamma, \mathbf{N}) := \{\rho \in R_0(\Gamma, \mathbf{N}) \mid |\rho(\xi)| \leqslant \varepsilon_1, |\nabla_\Gamma \rho(\xi)| \leqslant \varepsilon_1, \forall \xi \in \Gamma\}.$$

The proof of the above proposition is based on the following lemma.

Lemma 3

Let $k \in \mathbb{N}$, $\alpha \in [0,1)$, and $\Omega \subset \mathbb{R}^d$ be an open bounded set with $C^{k+\alpha}$ boundary. Let $\phi \in C_0^{k+\alpha}(\Omega)$ and consider $\varphi(x) = x + \phi(x)$, $x \in \Omega$. Assume that $\max_{x\in\overline{\Omega}} \|\nabla^\top \phi(x)\| < 1$. Then, $\det(\nabla^\top \varphi) > 0$ and $\varphi \in \mathrm{Diffeo}^{k+\alpha}(\Omega, \Omega)$; i.e., the map $\varphi : \Omega \to \Omega$ is a $C^{k+\alpha}$-diffeomorphism.

Preliminaries: a quasi-stationary moving boundary problem

For $\rho \in R_{[a,b]}(\Gamma, \mathbf{N})$, we define the moving boundary

(2) $$\mathcal{M}(\rho, [a,b]) := \bigcup_{t \in [a,b]} \mathcal{S}(\rho(t)) \times \{t\},$$

with normal velocity $V_n(t) = V_n(\cdot, t) \in C^0(\mathcal{S}(\rho(t)))$ where

$$V_n(x,t) := \rho_t(\xi, t)\mathbf{N}(\xi) \cdot \nu(x; \mathcal{S}(\rho(t))), \qquad x = \xi + \rho(\xi,t)\mathbf{N}(\xi) \in \mathcal{S}(\rho(t)), \quad \xi \in \Gamma.$$

We define the set of moving boundaries

$$M_{[a,b]}(\Gamma, \mathbf{N}) := \left\{ \mathcal{M}(\rho, [a,b]) \subset \mathbb{R}^d \times \mathbb{R} \mid {}^{\exists}\rho \in R_{[a,b]}(\Gamma, \mathbf{N}) \text{ such that (2) is satisfied} \right\}.$$

Problem 4

Given $\Gamma_\circ \in \mathcal{A}^{2+\alpha}$, $f \in C^{2+\alpha}(\Sigma)$, *and* $g \in C^{1+\alpha}(\Sigma)$, *find* $T > 0$ *and* $\mathcal{M} = \bigcup_{0 \leqslant t \leqslant T} \Gamma(t) \times \{t\}$ *such that*

(3) $$\begin{cases} V_n(t) = -\left[\dfrac{\partial}{\partial \nu} u_D(\Gamma(t)) - \dfrac{\partial}{\partial \nu} u_N(\Gamma(t)) \right] & \text{on } \Gamma(t),\ (0 \leqslant t \leqslant T), \\ \Gamma(0) = \Gamma_\circ, & \end{cases}$$

where $u_D(\Gamma(t))$ *and* $u_N(\Gamma(t))$ *are defined by* (D) *and* (N).

Definition 5

We say $\mathcal{M} = \bigcup_{0 \leqslant t \leqslant T} \Gamma(t) \times \{t\} \subset \mathbb{R}^d \times \mathbb{R}$ **a solution of Problem 4**, if for $\Gamma(0) = \Gamma_0$, there exists a collection of closed intervals $\{I_k\}_{k=1}^n$ such that $\bigcup_{k=1}^n I_k = [0, T]$, and for each k, there exists $t_k \in I_k$, $\Gamma_k \in \mathcal{A}^{2+\alpha}$, and quasi-normal $\mathbf{N}_k$ on Γ_k such that

$$\mathcal{M}\big|_{I_k} \in M_{I_k}(\Gamma_k, \mathbf{N}_k) \quad \text{where} \quad \mathcal{M}\big|_{I_k} = \bigcup_{t \in I_k} \Gamma(t) \times \{t\},$$

is a solution of

$$V_n(t) = -\left[\frac{\partial}{\partial \nu} u_{\mathrm{D}}(\Gamma(t)) - \frac{\partial}{\partial \nu} u_{\mathrm{N}}(\Gamma(t))\right] \text{ on } \Gamma(t) \text{ for } t \in I_k, \text{ for each } k = 1, \ldots, n,$$

where $u_{\mathrm{D}}(\Gamma(t))$ and $u_{\mathrm{N}}(\Gamma(t))$ are defined by (D) and (N), respectively.

Remark 2

*We note that the definition of V_n **does not depends on the choice of Γ_k and $\mathbf{N}_k$.***

Question: *suppose* $\mathcal{M}_{[0,T]}$ *is a solution to* (HSP)*, then is it true that* $\mathcal{M}_{[t_*,T]}$ *is also a solution to* (HSP) *for* $t_* \in [0,T]$*?*

The next lemma answers this question affirmatively.

Lemma 6

Let

- $\Gamma \in \mathcal{A}^{2+\alpha}$,
- $\mathbf{N}$ *be a quasi-normal vector on* Γ,
- $\mathcal{M} = \bigcup_{a \leqslant t \leqslant b} \Gamma(t) \times \{t\} \in M_{[a,b]}(\Gamma, \mathbf{N})$,
- $t_* \in [a,b]$, *and*
- $\mathbf{N}_*$ *be quasi-normal on* $\Gamma(t_*)$.

Then, there exists a $\delta > 0$ *such that, setting* $I_* := [a,b] \cap [t_* - \delta, t_* + \delta]$, *we have*

$$\mathcal{M}\big|_{I_*} \in M_{I_*}(\Gamma(t_*), \mathbf{N}_*). \tag{4}$$

Definition 7

We say $\mathcal{M} = \bigcup_{0\leqslant t\leqslant T} \Gamma(t) \times \{t\} \subset \mathbb{R}^d \times \mathbb{R}$ **a solution of Problem 4**, if for all $t_* \in [0,T]$, there exist $a < b$ and $\delta > 0$ such that $[t_* - \delta, t_* + \delta] \cap [0,T] \subset [a,b] \subset [0,T]$ and there exists a quasi-normal vector $\mathbf{N}_*$ on $\Gamma(t_*)$ such that

$$\mathcal{M}_{[a,b]} \in M_{[a,b]}(\Gamma(t_*), \mathbf{N}_*) \quad \text{where} \quad \mathcal{M}_{[a,b]} := \bigcup_{a\leqslant t\leqslant b} \Gamma(t) \times \{t\},$$

and $u_{\mathsf{D}}(\Gamma(t))$ and $u_{\mathsf{N}}(\Gamma(t))$ defined by (D) and (N) solve Problem 4.

Remark By the previous lemma, observe that for all $t \in [0,T]$, there exists a $\delta(t) > 0$ such that $[t - \delta(t), t + \delta(t)] \cap [0,T] \subset [a,b] \subset [0,T]$, and there exists a quasi-normal vector $\mathbf{N}$ on $\Gamma(t)$ such that $\mathcal{M}_{[a,b]} \in M_{[a,b]}(\Gamma(t), \mathbf{N})$, and $u_{\mathsf{D}}(\Gamma(t))$ and $u_{\mathsf{N}}(\Gamma(t))$ defined by (D) and (N) solve (3). Now, we observe that

$$\emptyset \neq \mathcal{O}(t) := \begin{cases} (t - \delta(t), t + \delta(t)) \cap (0,T) & \text{for } t \in (0,T), \\ [0, \delta(0)) & \text{for } t = 0, \\ (t - \delta(t), T] & \text{for } t = T. \end{cases}$$

Note that $\mathcal{O}$ is open in $[0,T]$ and $\bigcup_{t\in[0,T]} \mathcal{O}(t) = [0,T]$.

Notations (1/2)

For $l \in \mathbb{R}_+$, $C^0([0,T];C^l(\overline{\Omega}))$ denotes the space of continuous functions with respect to

$$(x,t) \in \left\{(x,t) \mid t \in [0,T], x \in \overline{\Omega}\right\}$$

with the finite norm

$$\max_{0 \leqslant t \leqslant T} |u(\cdot,t)|_{\overline{\Omega}}^{(l)},$$

where

$$|u|_{\overline{\Omega}}^{(l)} :=_{C^{[l],l-[l]}(\overline{\Omega})} = |u|_{[l],l-[l];\overline{\Omega}} = \sum_{|j|<l} \max_{\overline{\Omega}} |D^j u(x)| + [u]_{\overline{\Omega}}^{(l)},$$

$$[u]_{\overline{\Omega}}^{(l)} := [u]_{[l],l-[l];\overline{\Omega}} = \sum_{|j|=[l]} \max_{x,\hat{x} \in \overline{\Omega}} \frac{|D^j u(x) - D^j u(\hat{x})|}{|x-\hat{x}|^{l-[l]}}.$$

The spaces $C^0([0,T];C^l(\Sigma))$ and $C^0([0,T];C^l(\Gamma))$ are introduced in a similar manner.

For pair of functions φ_{D} and φ_{N}, we will extensively use the following special notations (or operators):

$$\varphi_{\mathsf{DN}} := \varphi_{\mathsf{D}} - \varphi_{\mathsf{N}} \quad \text{and} \quad \varphi_{\mathsf{ND}} := \varphi_{\mathsf{N}} - \varphi_{\mathsf{D}}.$$

For example, we write $u_{\mathsf{DN}} = u_{\mathsf{D}} - u_{\mathsf{N}}$ and $u_{\mathsf{DN}}(\Gamma) = u_{\mathsf{D}}(\Gamma) - u_{\mathsf{N}}(\Gamma)$.

Let a, b be fixed real numbers such that $b > a$, $k \in \mathbb{N} \cup \{0\}$, $\alpha \in (0,1)$, and $\Xi \in \{\overline{\Omega}, \Gamma, \Sigma\}$. For well-defined functions $\varphi, u, v, f, g, \rho$, etc, we introduce the following norms for economy of space:

$$|\varphi|^{(k+\alpha)}_{\Xi;\,[a,b]} := \max_{a \leqslant \tau \leqslant b} |\varphi(\cdot,\tau)|^{(k+\alpha)}_{\Xi}\,,$$

$$|\varphi|^{\infty}_{\Xi;\,[a,b]} := \max_{a \leqslant \tau \leqslant b} \max_{\Xi} |\varphi(\cdot,\tau)|\,,$$

$$\|(u,v)\|^{(k+\alpha)}_{\Xi;\,[a,b]} := |u|^{(k+\alpha)}_{\Xi;\,[a,b]} + |v|^{(k+\alpha)}_{\Xi;\,[a,b]} = \max_{a \leqslant t \leqslant b} |u(\cdot,\tau)|^{(k+\alpha)}_{\Xi} + \max_{a \leqslant t \leqslant b} |v(\cdot,\tau)|^{(k+\alpha)}_{\Xi}\,,$$

$$\|\varphi_{\mathsf{D,N}}\|^{(k+\alpha)}_{\Xi;\,[a,b]} := \|(\varphi_{\mathsf{D}}, \varphi_{\mathsf{N}})\|^{(k+\alpha)}_{\Xi;\,[a,b]}\,,$$

$$\|(f,g)\|^{(2+\alpha)}_{\Sigma;\,[a,b]} := \max_{a \leqslant \tau \leqslant b} |f(\cdot,\tau)|^{(2+\alpha)}_{\Sigma} + \max_{a \leqslant \tau \leqslant b} |g(\cdot,\tau)|^{(1+\alpha)}_{\Sigma}\,, \qquad |||(f,g)||| := \|(f,g)\|^{(2+\alpha)}_{\Sigma;\,[0,t]}\,,$$

$$|||\rho|||^{(k+\alpha)}_{\Xi;\,[a,b]} := \max_{a \leqslant \tau \leqslant b} |\rho(\cdot,\tau)|^{(k+\alpha)}_{\Xi} + \max_{a \leqslant \tau \leqslant b} \left| \frac{d}{d\tau}\rho(\cdot,\tau) \right|^{(k-1+\alpha)}_{\Xi}\,,$$

$$|||\rho|||^{(k+\alpha)}_{\overline{\Omega},\,\Gamma;\,[a,b]} := \max_{a \leqslant \tau \leqslant b} |\rho(\cdot,\tau)|^{(k+\alpha)}_{\overline{\Omega}} + \max_{a \leqslant \tau \leqslant b} \left| \frac{d}{d\tau}\rho(\cdot,\tau) \right|^{(k-1+\alpha)}_{\Gamma}\,.$$

Theorem 8

Let the following assumption be satisfied:

(A1) *For some* $\alpha \in (0,1)$,

$$\Sigma,\, \Gamma = \Gamma_\circ \in C^{2+\alpha}, \quad f \in C^0([0,T]; C^{2+\alpha}(\Sigma)),\, f > 0, \quad g \in C^0([0,T]; C^{1+\alpha}(\Sigma)),\, g > 0,$$

such that

$$\frac{\partial}{\partial \nu}(u_{DN}(\Gamma_\circ)) > 0,$$

where u_D *and* u_N *respectively solves* (D) *and* (N) *in* $\Omega(\Gamma_\circ)$.

Then, there exists a unique solution $\Gamma(t)$, $u_D(x,t)$, *and* $u_N(x,t)$ *to* (HSP) *defined on some small time-interval* $I^\star = [0, t^\star]$, *where* $t^\star < T$.

Theorem 8 (continuation)

The free surface $\Gamma(t)$ is described by the equation

(5) $$x = \xi + \rho(\xi, t)\mathbf{N}(\xi), \qquad \xi \in \Gamma,$$

where ξ is the local coordinate on the surface Γ and $\mathbf{N}$ is a smooth vector field on Γ such that $\mathbf{N} \cdot \nu_\circ \geqslant \nu_\star > 0$, where $\nu_\circ$ is the unit normal vector to the surface Γ directed *inward* the domain $\Omega(\Gamma)$.

The function $\rho \in C^0(I^\star; C^{2+\alpha}(\Gamma))$ has extra smoothness with respect to the variable t; namely, $\rho_t \in C^0(I^\star; C^{1+\alpha}(\Gamma))$. Meanwhile, the functions $u_{\mathsf{D}}(x,t)$ and $u_{\mathsf{N}}(x,t)$ are defined in $\Omega(t)$ for $t \in I^\star$ and both belong to the space $C^0(I^\star; C^{2+\alpha}(\overline{\Omega(t)}))$.

Moreover, the following estimate hold

(6) $$\|u_{\mathsf{D,N}}\|^{(2+\alpha)}_{\overline{\Omega};\,[0,t]} + \|\rho\|^{(2+\alpha)}_{\Gamma;\,[0,t]} \leqslant c\,\|(f,g)\|^{(2+\alpha)}_{\Sigma;\,[0,t]} \leqslant c\,|\!|\!|(f,g)|\!|\!|.$$

for some constant $c > 0$, for all $t \in I^\star$.

Lemma 9

Let $\Omega \subset \mathbb{R}^d$, of class $C^{2+\alpha}$, be an open bounded connected set with non-intersecting boundaries Γ and Σ. Assume that $v \in C^{2+\alpha}(\overline{\Omega}) \cap C^{0+\alpha}(\overline{\Omega})$ and

$$-\Delta v = 0 \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \Gamma, \qquad \frac{\partial v}{\partial \nu} > 0 \quad \text{on } \Sigma.$$

Then, $v > 0$ in Ω.

Proposition 3

Let $\Omega = D \setminus \overline{\omega} \subset \mathbb{R}^d$, of class $C^{2+\alpha}$, be an open bounded connected set with non-intersecting boundaries $\partial\omega = \Gamma \in \mathcal{A}^{2+\alpha}$ and $\Sigma = \partial D$. Assume that $\partial\omega^\star = \Gamma^\star \in \mathcal{A}^{2+\alpha}$ is the exact interior boundary that satisfies (IP) *and ω strictly contained $\overline{\omega}^\star$ (i.e., Γ lies entirely in the interior of $\overline{\Omega}^\star = \overline{D \setminus \overline{\omega}^\star}$). Let $f \in C^{2+\alpha}(\Sigma)$ and $g \in C^{1+\alpha}(\Sigma)$. Then, the functions $u_D(\Gamma)$ and $u_D(\Gamma)$ satisfying* (D) *and* (N)*, respectively, satisfy the following condition*

$$u_D > u_N \quad \text{in } \Omega.$$

Consequently,

$$\frac{\partial}{\partial \nu}(u_D - u_N) > 0 \quad \text{on } \Gamma.$$

Given the short-time existence of solution to (HSP), we can also prove the uniqueness of solution to the system.

Theorem 10

A solution of Problem 4 is unique.

Proof.

Assume that $\mathcal{M}_i := \bigcup \Gamma^i(t) \times \{t\}$, $i = 1, 2$, solves Problem 4, We suppose $\mathcal{M}_1 \neq \mathcal{M}_2$. Then, there exists $t_* \in [0, T)$ and a sequence $\{t_k\}_{k=1}^{\infty} \in (t_*, T]$ such that

$$\begin{cases} \mathcal{M}_1\big|_{[0,t_*]} = \mathcal{M}_2\big|_{[0,t_*]}, \\ T \geqslant t_1 > t_2 > \cdots > t_*, \quad \text{where } \lim_{k\to\infty} t_k = t_*, \text{ and} \\ \Gamma^1(t_k) \neq \Gamma^2(t_k), \quad \text{for } k = 1, 2, \ldots. \end{cases} \tag{7}$$

Since $\Gamma(t_*) := \Gamma^1(t_*) = \Gamma^2(t_*)$ satisfies the conditions in Theorem 8, there exists $t_{**} \in (t_*, T]$ such that there is a unique $\Gamma(t)$ for $t \in [t_*, t_{**}]$. This contradicts the last two lines in (7). Thus, $\mathcal{M}_1$ and $\mathcal{M}_2$. □

Sketch of Proof of the Main Result

To prove the main result, we proceed with the following main steps:

Step 1. First, we reformulate the problem onto a fixed domain, and we establish the classical solvability of the state problems on the fixed domain.

Step 2. Then, we separate the linear components of the primary and dynamic boundary conditions of the nonlinear problem from Step 1, placing all nonlinear components on the left-hand side of the resulting equation.

Step 3. Following that, we demonstrate the existence of a classical solution to the linear problem associated with the system from Step 2 and obtain a key estimate for the solutions. The approach involves using the method of successive approximations or the Schauder method (refer to [GT01, p. 74] or [Vol14, Sec. 1.1.1, p. 124]).

Step 4. We then establish the uniqueness of the solution by comparing two solutions and proving they are identical using the estimate from Step 3.

Step 5. Finally, by utilizing certain interpolation inequalities and the classical solution to the linear problem proven in Step 3, we demonstrate the short-time existence of a classical solution to the nonlinear problem derived from Step 1 through the method of successive approximations. The proof concludes by transforming the fixed domain back to the moving domain.

Summary

- We revisited a shape optimization reformulation of a shape inverse problem and proposed an efficient numerical approach for solving it.
- Additionally, we studied the existence, uniqueness, and continuous dependence of a classical solution to a Hele-Shaw-like system derived from this formulation.
- We reiterate that little to no work has been done with respect to the well-posedness of the shape optimization problem related to the system studied here, specifically in the present research direction.
- The system examined in this study is novel. Hence, the analysis carried out in this work, inspired by Bizhanova and Solonnikov, offers a fresh perspective.
- We anticipate that the same analysis could be applied to other Hele-Shaw-like systems resulting from shape optimization reformulations of a shape identification problem.

Thank you for your kind attention.

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