Extension of the entropy dissipation method to inhomogeneous non-linear Fokker-Planck equations

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6th June, 2024@Hilton Niseko Village,

The 81st Fujihara seminar, “Mathematical Aspects for Interfaces and Free Boundaries”

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Inhomogeneous non-linear Fokker-Planck equations

$$\begin{cases} f_t - \operatorname{div}\left(f\nabla\left(D(x)\log f + \phi(x)\right)\right) = 0, & x \in \Omega,\ t > 0, \\ f(x,0) = f_0(x), & x \in \Omega. \end{cases} \tag{FP}$$

- $\Omega = [0,1)^n$ with the periodic boundary condition for f.
- $D = D(x) : \Omega \to \mathbb{R}$: given positive periodic smooth function.
- $\phi = \phi(x) : \Omega \to \mathbb{R}$: given periodic smooth function.
- $f_0 = f_0(x) : \Omega \to \mathbb{R}$: given periodic smooth probability density function.
- $f = f(x,t) : \Omega \times [0,\infty)$: unknown probability density function (PDF).

We want to know...

A condition that a classical solution f of (FP) converges to the equilibrium state $f_{\mathrm{eq}}(x) = \exp\left(-\frac{\phi(x)-C}{D(x)}\right)$ as $t \to \infty$.

Main talk: Why do we consider (FP)?

Motivation: Why (FP)?

Consider the following Stochastic differential equation

$$\frac{dx}{dt} = -\nabla\phi(x) + \sqrt{2D}\frac{dB}{dt}, \tag{SDE}$$

where B is a Brownian motion/Wiener Process, D is a positive coefficient.

If D is a constant, the associated PDF f of (SDE) satisfies

$$f_t = \Delta(Df) + \mathrm{div}(f\nabla\phi(x)) = \mathrm{div}(f\nabla(D\log f + \phi(x))).$$

Then, we can deduce the following energy law (I explain later):

$$\frac{d}{dt}\int_\Omega (Df(\log f - 1) + f\phi(x))\,dx = -\int_\Omega |\nabla(D\log f + \phi(x))|^2 f\,dx$$

NOTE $-\nabla(D\log f + \phi(x))$ is called velocity in the continuity equation.

Motivation: Why (FP)?

Consider the following Stochastic differential equation

$$\frac{dx}{dt} = -\nabla\phi(x) + \sqrt{2D}\frac{dB}{dt} \qquad \text{(SDE)}$$

where B is a Brownian motion/Wiener Process, D is a positive coefficient.

If D is a function of x, we should determine/point out the stochastic integration (Ito, Stratonovich, etc)).

ITO $f_t = \Delta(D(x)f) + \mathrm{div}(f\nabla\phi(x))$

Stratonovich $f_t = \mathrm{div}(\sqrt{D(x)}\nabla(\sqrt{D(x)}f)) + \mathrm{div}(f\nabla\phi(x))$

backward $f_t = \mathrm{div}(D(x)\nabla f) + \mathrm{div}(f\nabla\phi(x))$

It is difficult to deduce the energy law:

$$\frac{d}{dt}\int_\Omega \text{(What is the form?)}\, dx = -\int_\Omega |\text{velocity}|^2 f\, dx$$

Motivation: Why (FP)?

Returning to the energy law with the constant D:

$$\begin{aligned}\frac{d}{dt}\int_\Omega (Df(\log f - 1) + f\phi(x))\,dx &= \int_\Omega (D\log f + \phi(x)) f_t\,dx \\ &= -\int_\Omega (D\log f + \phi(x))\,\mathrm{div}(fu)\,dx \\ &= -\int_\Omega f|u|^2\,dx\end{aligned}$$

where $u = -\nabla(D\log f + \phi(x))$. We can proceed the above computation even though D is a function of x. Namely, to guarantee

$$\frac{d}{dt}\int_\Omega (D(x)f(\log f - 1) + f\phi(x))\,dx = -\int_\Omega f|u|^2\,dx \qquad \text{(EnergyLaw)}$$

along with the continuity equation $f_t + \mathrm{div}(fu) = 0$, we may find $u = -\nabla(D(x)\log f + \phi(x))$.

Motivation: Why (FP)?

Why consider variable coefficients? To answer it, consider (SDE) again:

$$\frac{dx}{dt} = -\nabla\phi(x) + \sqrt{2D}\frac{dB}{dt} \tag{SDE}$$

Here D is the coefficient of the Brownian motion.

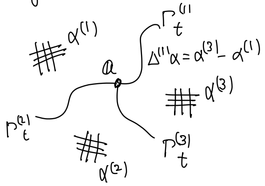
- D is related to other parameters/variables, like temperature, etc.
- To understand multi-scale models, energy law is important.
- From the SDE, it is difficult to construct the energy law.

Our approach to derive (FP) is based on the (EnergyLaw)

$$\frac{d}{dt}\int_\Omega (D(x)f(\log f - 1) + f\phi(x))\,dx = -\int_\Omega f|u|^2\,dx \tag{EnergyLaw}$$

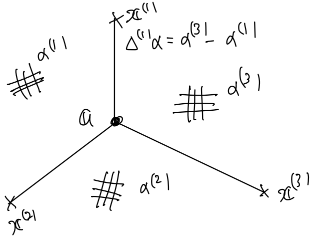
Motivation: Relationship between evolution of grain boundaries and (FP)

⟨Original model⟩



$$E = \sum_{j=1}^{3} \int_{\Gamma_t^{(j)}} \sigma(\Delta^{(j)}\alpha(t), n(s,t))\,ds$$

⟨Reduced model⟩ (Relaxation to the curvature effect)



$$E = \sum_{j=1}^{3} \sigma(\Delta^{(j)}\alpha(t))\,|a(t) - x^{(j)}|$$

In SIAM MA(2021), and CMS(2021), we considered the following model:

$$\frac{d(\Delta\alpha)}{dt} = -3\gamma \nabla_{\Delta\alpha} E, \qquad \frac{da}{dt} = -\eta \nabla_a E \qquad \text{(GBM)}$$

where $\Delta\alpha = (\Delta^{(j)}\alpha)$ is a misorientation, a is a triple junction, and

$$E(\Delta\alpha, a) = \sum_{j=1}^{3} \sigma(\Delta^{(j)}\alpha)|a - x_j|.$$

In M3AS(2022), to understand the interaction between the misorientations and the triple junction of numerous grain boundaries, we added the white noise:

$$\frac{d(\Delta\alpha)}{dt} = -3\gamma \nabla_{\Delta\alpha} E + \beta_{\Delta\alpha}\frac{dB}{dt}, \qquad \frac{da}{dt} = -\eta \nabla_a E + \beta_a \frac{dB}{dt}. \quad \text{(GBM-SDE)}$$

For constants $\beta_{\Delta\alpha}, \beta_a$, we derive a sufficient condition to guarantee the existence of the equilibrium state, and study long-time asymptotic behavior for PDF of (GBM-SDE).

$$\begin{cases} f_t - \operatorname{div}\left(f \nabla (D(x) \log f + \phi(x))\right) = 0, & x \in \Omega,\ t > 0, \\ f(x,0) = f_0(x), & x \in \Omega. \end{cases} \tag{FP}$$

Theorem (arXiv:2404.05157)

Assume $n = 1, 2, 3$. For a fixed constant $\gamma > 0$, there exist positive constants $C_1 > 0$, $C_2 > 0$ such that if

$$D(x) \geqq C_1, \qquad \int_\Omega |\nabla (D(x) \log f_0 + \phi(x))|^2 f_0 \, dx \leqq C_2,$$

then there is $C > 0$ such that for all $t > 0$

$$\int_\Omega |\nabla (D(x) \log f + \phi(x))|^2 f \, dx \leqq C e^{-\gamma t}.$$

Meaning: $\nabla (D(x) \log f + \phi(x)) \to 0 \Longrightarrow D(x) \log f + \phi(x) \to$ Const

Known results: Entropy dissipation methods

$$\begin{cases} f_t - \mathrm{div}\,(f\nabla(D(x)\log f + \phi(x))) = 0, & x \in \Omega,\ t > 0, \\ f(x,0) = f_0(x), & x \in \Omega. \end{cases} \tag{FP}$$

The entropy dissipation method is well-known to explore asymptotic behavior for the Fokker-Planck equation. For the constant D case,

- Carrillo-Toscani, Math. Methods Appl. 1998.
- Markowich-Villani, Proceeding of IV Workshop on PDEs, 2000.
- Arnold-Markowich-Toscani-Unterreiter, CPDE, 2001
- Jüngel, SpringerBriefs in Math., 2016.

They studied L^1 convergence by using the Csiszár-Kullback-Pinsker inequality.

There are not so much results for the variable diffusion coefficient case. For example, Arnold-Markowich-Toscani-Unterreiter studied

$$f_t = \mathrm{div}(D(x)(\nabla u + u\nabla\phi(x))).$$

Key idea of proof: Extension to the entropy dissipation method

$$\begin{cases} f_t - \operatorname{div}\left(f \nabla (D(x) \log f + \phi(x))\right) = 0, & x \in \Omega,\ t > 0, \\ f(x,0) = f_0(x), & x \in \Omega. \end{cases} \tag{FP}$$

$$\frac{d}{dt} \int_\Omega (D(x) f (\log f - 1) + f\phi(x))\, dx = - \int_\Omega |\nabla (D(x) \log f + \phi(x))|^2 f\, dx.$$

Compute 2nd derivative of the energy ($u = -\nabla (D(x) \log f + \phi(x))$)

$$\begin{aligned} &\frac{d^2}{dt^2} \int_\Omega (D f (\log f - 1) + f\phi(x))\, dx \\ &= 2 \int_\Omega (\nabla^2 \phi(x) u \cdot u) f\, dx + 2 \int_\Omega D(x) |\nabla u|^2 f\, dx + \int_\Omega (\text{including } \nabla D)\, dx \\ &\geqq 2 \int_\Omega (\nabla^2 \phi(x) u \cdot u) f\, dx + 2 \int_\Omega D(x) |\nabla u|^2 f\, dx \\ &\quad - \frac{1}{\min D(x)} \int_\Omega D(x)\, |(\text{including } \nabla D)|\, dx. \end{aligned}$$

Why $n \leqq 3$? : (including $\nabla D(x)$) has $|u|^3$.