Uniqueness in inverse problem of determining shapes of sub-boundaries by nonstationary heat equations without initial conditions

Masahiro Yamamoto
The University of Tokyo,
Zonguldak Bülent Ecevit University

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Joint with

Professor A. Doubova (Universidad de Sevilla)

Professor E. Fernández-Cara (Universidad de Sevilla)

Professor J. Apraiz (Universidad del País Vasco, Leioa)

§1. Introduction

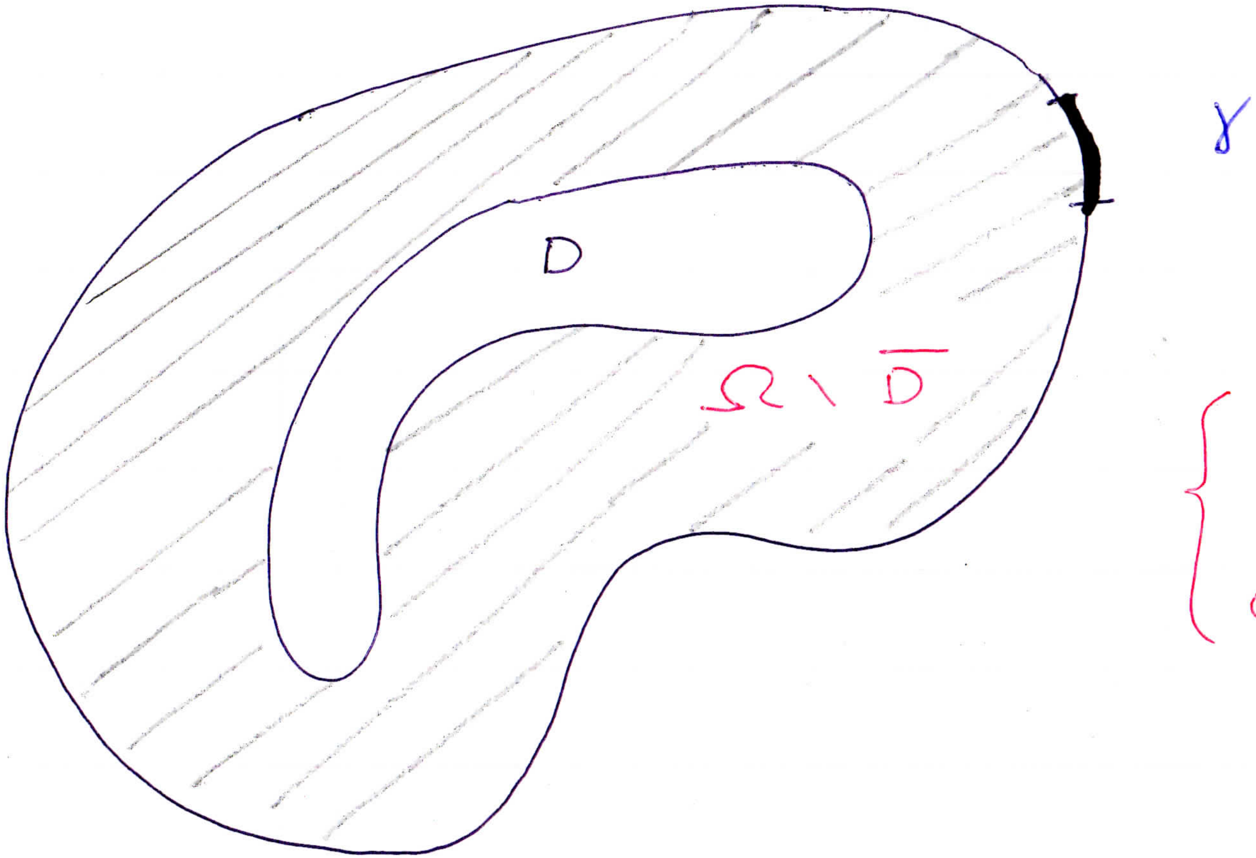
D: simply connected domain, $D \subset\subset \Omega \subset \mathbb{R}^n$.

$$\begin{cases} \partial_t u = \Delta u \quad \text{or} \quad \partial_t u = \text{div}\,(p(x)\nabla u) \quad \text{in } (\Omega \setminus \overline{D}) \times (0,T), \\ u = 0 \quad \text{or } \partial_\nu u = 0 \text{ on } \partial D \times (0,T) \end{cases}$$

Inverse problem: $u, \partial_\nu u$ on $\gamma \times (0,T) \Longrightarrow D$?

Here $\gamma \subset \partial\Omega$: accessible outer subboundary
∂D: inaccessible subboundary.

Main theoretical topic: uniqueness



$$\begin{cases} \partial_t u = \Delta u \\ \quad \text{in } \Omega \setminus \overline{D} \\ u = 0 \text{ on } \partial(\Omega \setminus \overline{D}) \end{cases}$$

Case with zero initial value: easy

Let $u \not\equiv 0$ *in* $(\Omega \setminus \overline{D}) \times (0,T)$. *Then* $(u, \partial_\nu u)|_{\gamma\times(0,T)} \Longrightarrow D$ *is 1 to 1.*

Proof. Let $D_1 \Rightarrow u$ and $D_2 \Rightarrow v$.

Sama Cauchy data $\Longrightarrow u = v$ in $(\Omega \setminus \overline{D_1 \cup D_2}) \times (0,T)$ by Unique Continuation

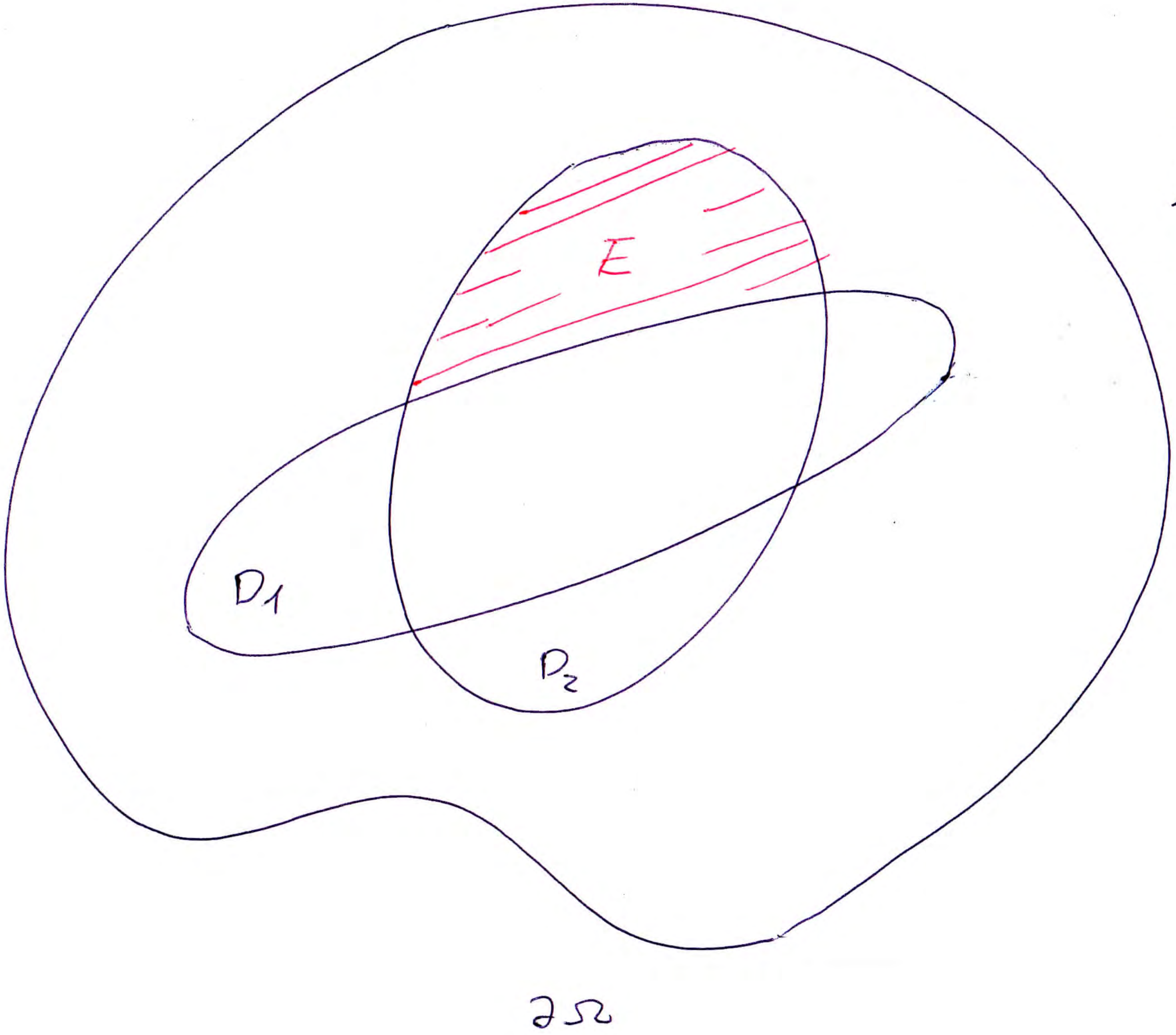
$v = 0$ on $\partial D_2 \Longrightarrow u = 0$ on $(\partial D_2 \setminus D_1) \times (0,T) \Longrightarrow$

$$\begin{cases} \partial_t u = \Delta u & \exists E \times (0,T), \\ u(\cdot, 0) = 0 & \text{in } E, \\ u|_{\partial E\times(0,T)} = 0 \end{cases}$$

$\Longrightarrow u = 0$ in $E \times (0,T)$

Unique continuation implies $u \equiv 0$ in $(\Omega \setminus \overline{D_1}) \times (0,T) \Longrightarrow$ contradiction ■

Unique continuation: *Let* $\partial_t u = -Au$ *in* $\widetilde{\Omega} \times (0,T)$. *If* $u|_{\omega\times(0,T)} = 0$ *or* $u = \partial_\nu u = 0$ *on* $\gamma \times (0,T)$ *with some subdomain* ω *and subboundary* γ, *then* $u = 0$ *in* $\widetilde{\Omega} \times (0,T)$.



$$\begin{cases} \partial_t u = \Delta u \quad \text{in } E \times (0, T) \\ u|_{\partial E} = 0 \end{cases}$$

Main formulation: Initial values are also unknown!
We are motivated e.g., by estimation of interior status of blast furnace.

Nobody does not remember initial temperature distribution
when blast-furnace started to be operated e.g., 20 years ago.

References.

- Bryan-Caudill, Jr., (1997): assuming the whole boundary condition.
- Apraiz - J. Cheng - Doubova - Fernández-Cara - Yamamoto:
  one-dimensional case by heat and wave equations (2022):

Our Approach

- (i) Asymptotic uniqueness. We have uniqueness by ”big” boundary inputs within resolution tolerance levels:
  ”Larger inputs $\Longrightarrow$ resolution level can be finer”.
  Key: extension inequality of solution to $\partial_t u = \Delta u$
  Two versions: (a) Harnack inequality
  (b) Quantitative unique continuation by Carleman estimate
- (ii) Uniqueness by Bang-bang input.

§2. Asymptotic uniqueness by Carleman estimate

Let $D_1, D_2 \subset\subset \Omega$,

$$\begin{cases} \partial_t u = \Delta u \quad \text{in } (\Omega \setminus \overline{D_1}) \times (0,T), \\ u|_{\partial D_1} = 0, \end{cases}$$

$$\begin{cases} \partial_t v = \Delta v \quad \text{in } (\Omega \setminus \overline{D_2}) \times (0,T), \\ v|_{\partial D_2} = 0. \end{cases}$$

Let $\gamma \subset \partial\Omega$. *If* $u = v$, $\partial_\nu u = \partial_\nu v$ *on* $\gamma \times (0,T)$, *then* $D_1 = D_2$?

Theorem 1 (asymptotic uniqueness)

Assume that $||u||_*$, $||v||_* \leq M_0$*: a priori boundedness by some* $C^{\ell,m}(\overline{\Omega} \times [0,T])$*-norm.*
Then for $\forall\delta > 0$*, there exist* $\exists T > 0$ *(large) and* $\exists R(\delta, M_0) > 0$ *such that*

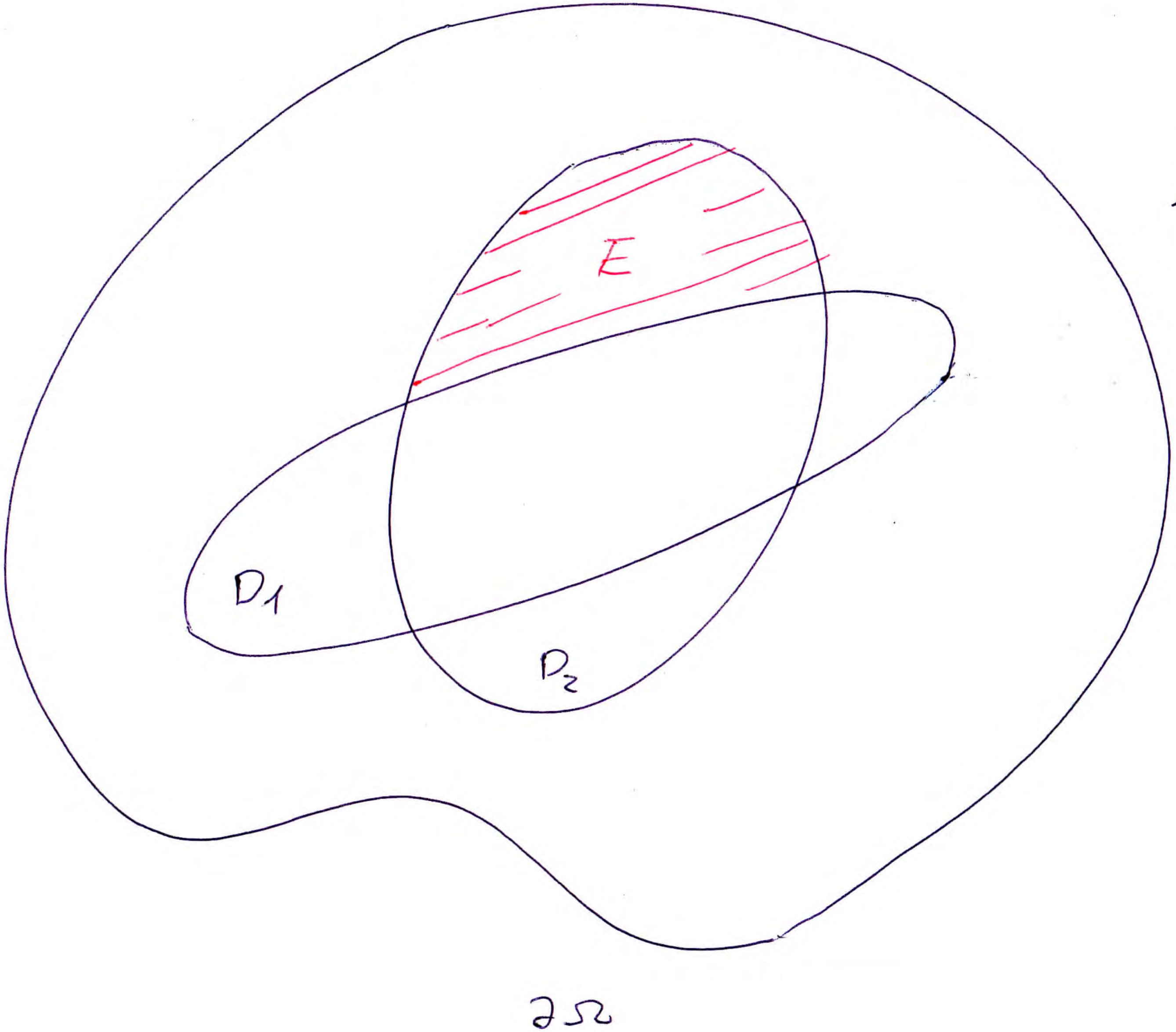
$$||u||_{L^2(\gamma\times(0,T))} > R(\delta, M_0)$$

implies

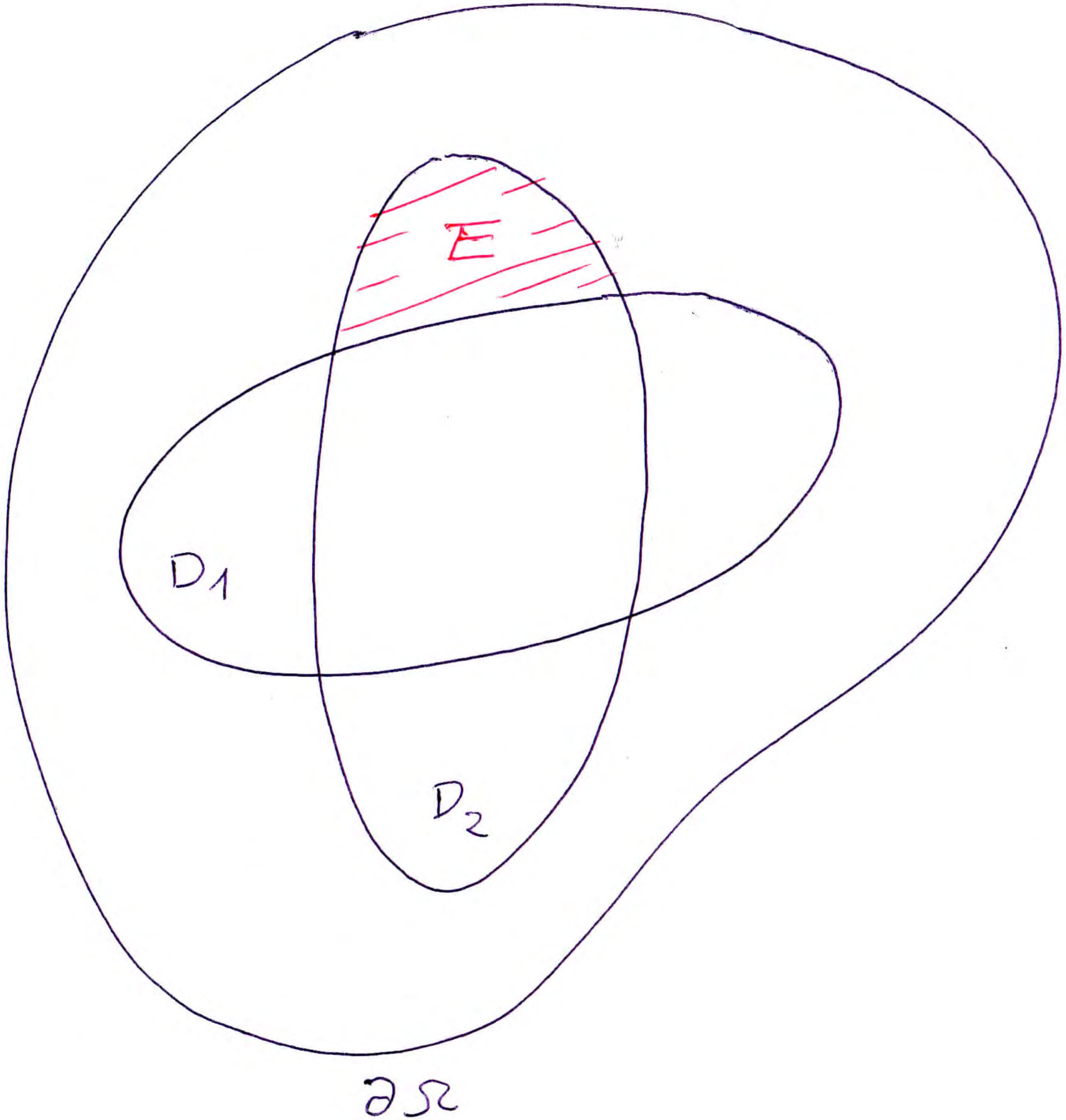
$$|(D_1 \setminus D_2) \cup (D_2 \setminus D_1)| < \delta.$$

Remark. *We can expect*

$$\lim_{\delta\downarrow 0} R(\delta, M_0) = \infty.$$



$$\begin{cases} \partial_t u = \Delta u \quad \text{in } E \times (0, T) \\ u|_{\partial E} = 0 \end{cases}$$



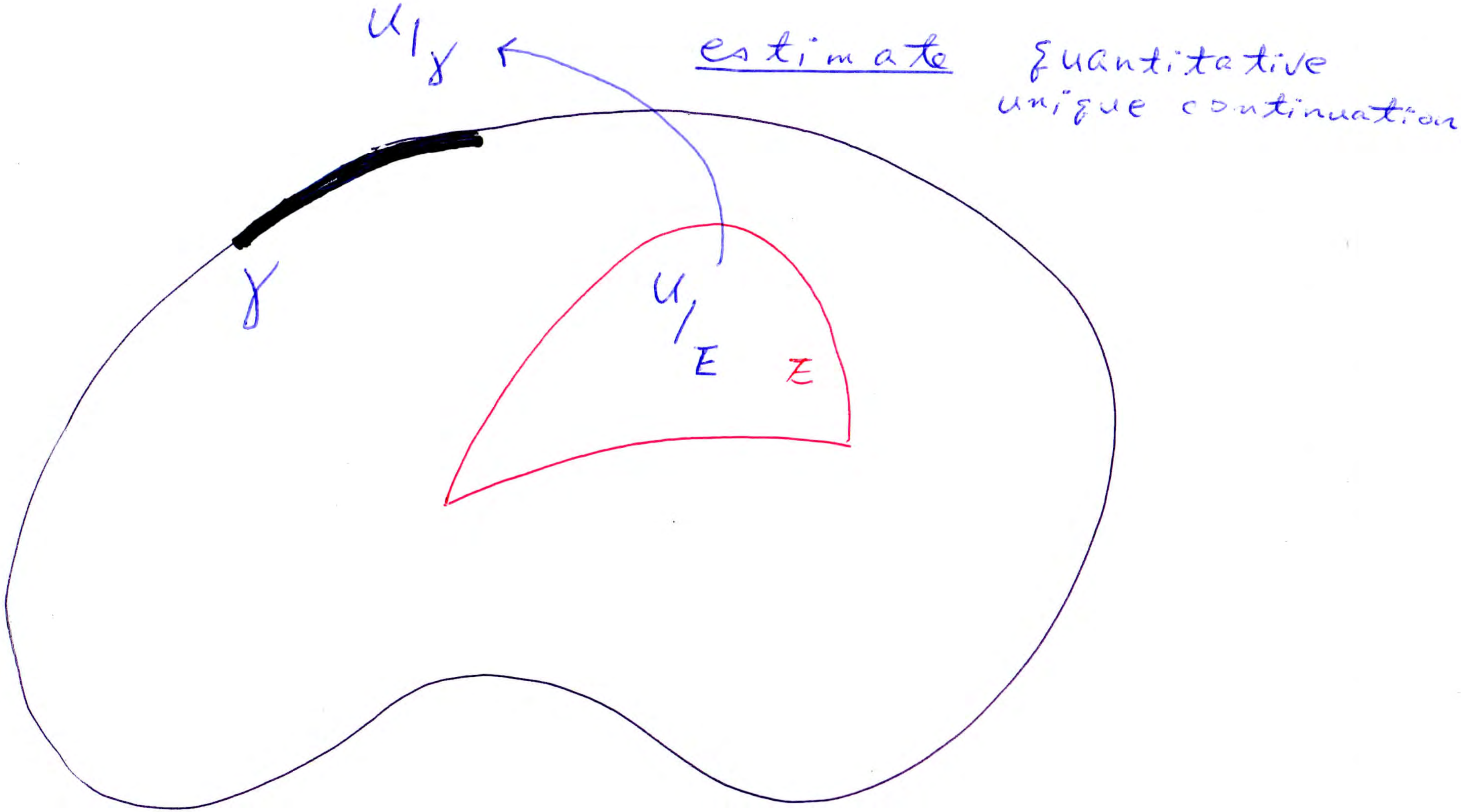
$$\begin{cases} \partial_t u = \Delta u & \text{in } E \times (0,T) \\ u|_{\partial E} = 0 \end{cases}$$

$$\Longrightarrow$$

$$\| u \|_{H^1(0,T;\, L^2(E))} \le M_0$$

if $\| u(\cdot, 0) \|_* \le M_0$

suitable norm



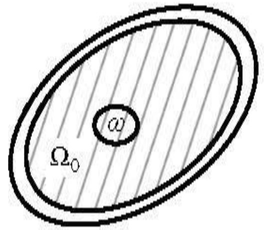
$u|_\gamma$: input large for M_0

$\Longrightarrow$ QED

Key:Quantitative unique continuation

Let $\Omega \subset \mathbb{R}^n$: bounded domain.

$$\begin{cases} \partial_t u = \Delta u \quad \text{in } \Omega \times (0,T), \\ \|u\|_{H^1(\partial\Omega\times(0,T))} + \|u\|_{L^\infty(0,T;H^1(\Omega))} \le M : \text{ arbitrarily given constant} \end{cases}$$



$\omega \subset\subset \Omega_0 \subset\subset \Omega$, $\varepsilon > 0$: given.

Then $\exists C > 0$, $\exists \theta \in (0,1)$ such that

$$\|u\|_{H^1(\varepsilon,T-\varepsilon;L^2(\Omega_0))} \le C\|u\|^{\theta}_{H^1(0,T;L^2(\omega))}.$$

Here C and θ depend on M, ω, Ω_0 and
are invariant under translations and rotations of coordinates.

M. Yamamoto, Introduction to Inverse Problems for Evolution Equations: Stability and Uniqueness by Carleman Estimates, to appear

Additional ingredient

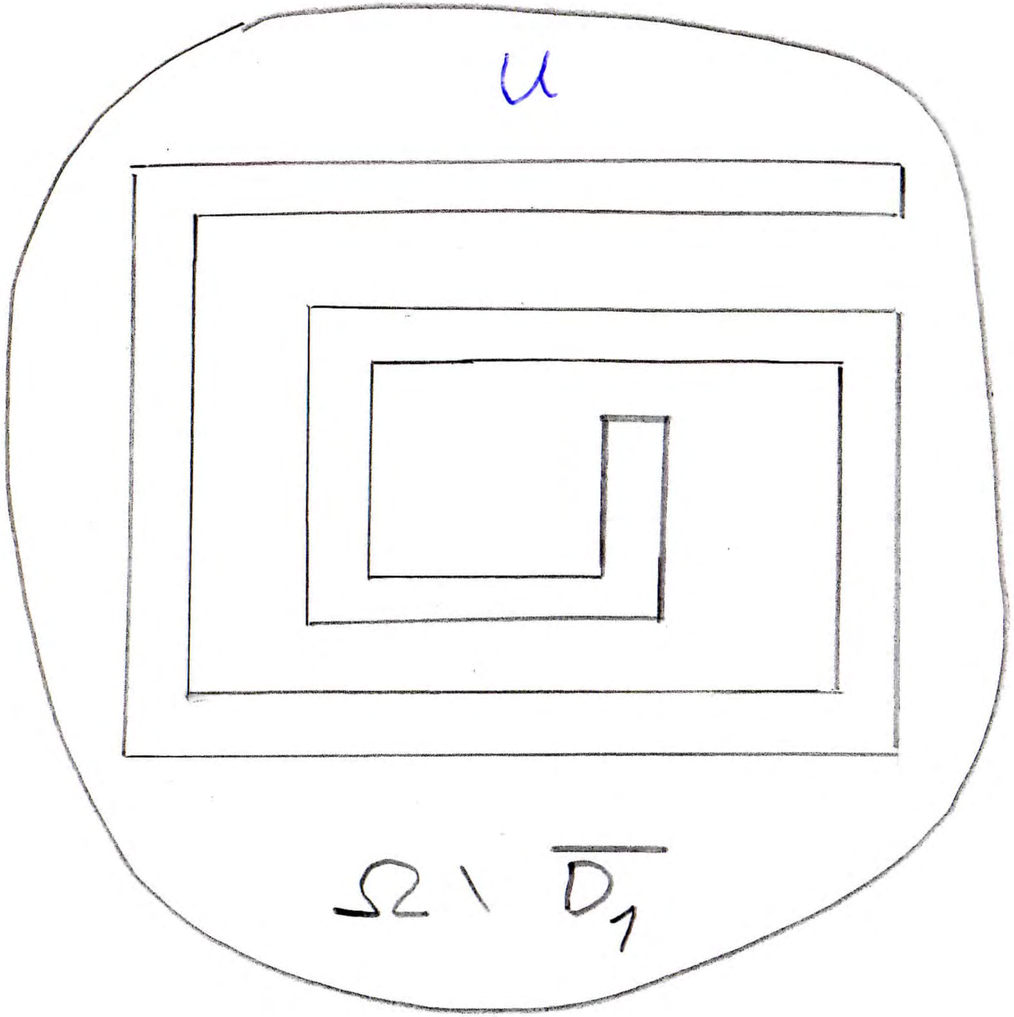
$$||u(\cdot, t)||_{H^m(E)} \leq Ce^{-c_0 t}$$

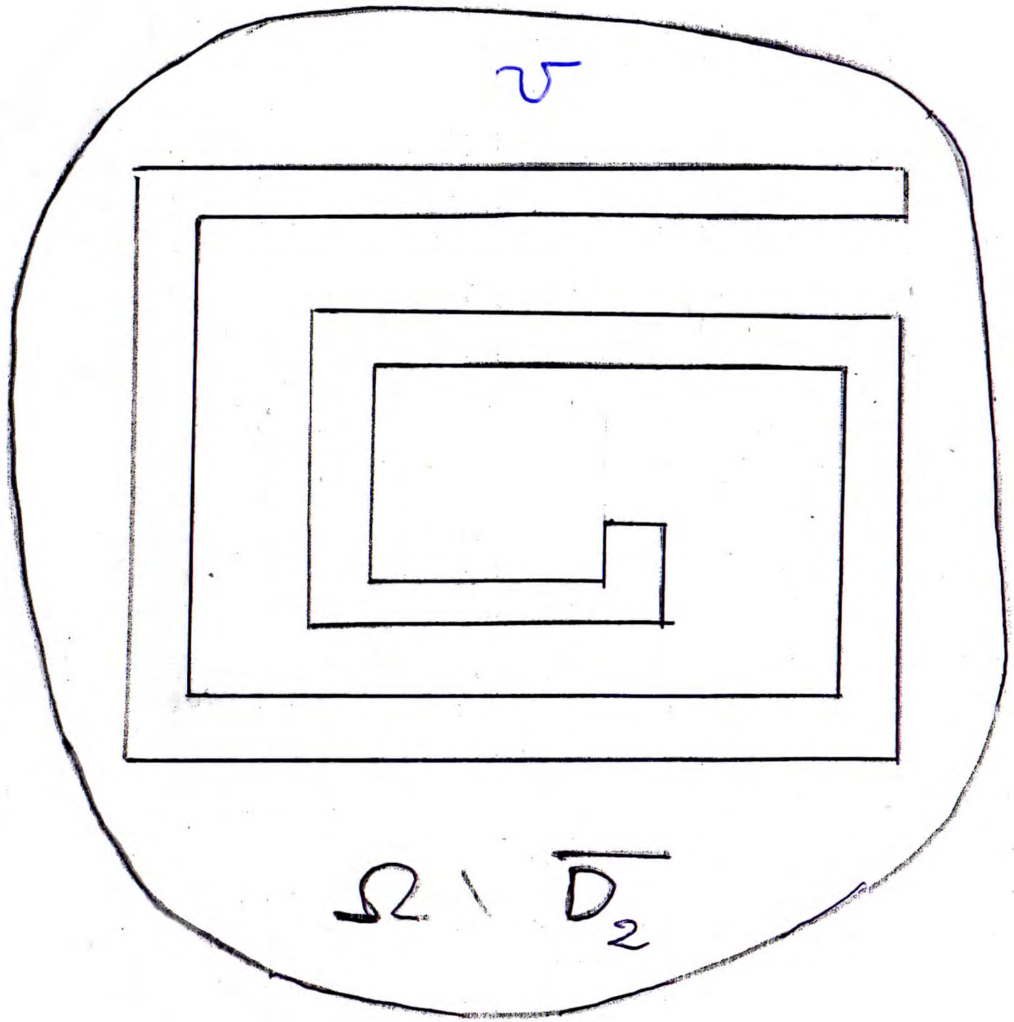
with constants C, c_0 depending on geometry of D.
$\Longrightarrow$ We can control amplitude of $u|_E$ also by choosing large $T > 0$.

$\Longrightarrow$

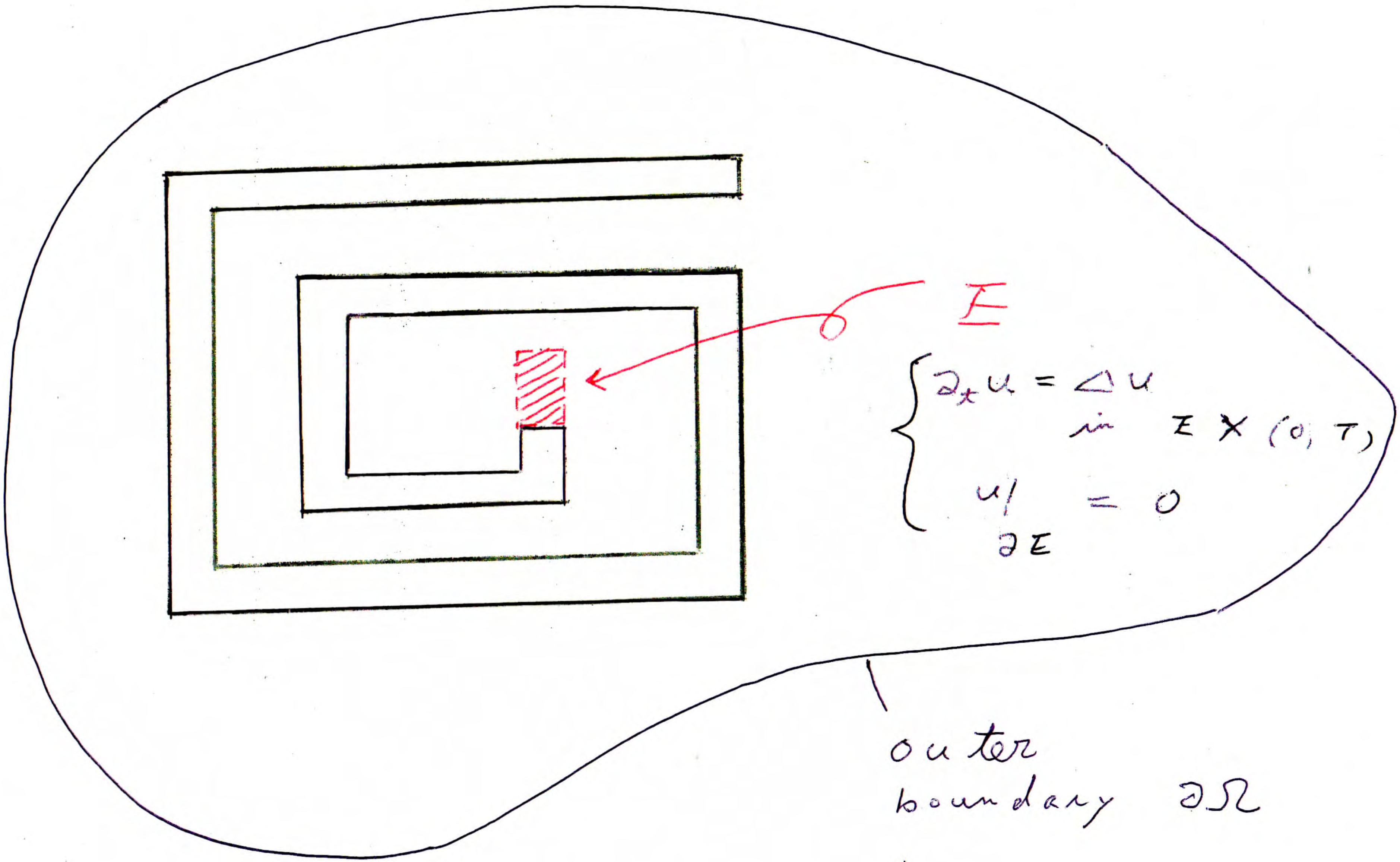
amplitude of boundary input $u|_{\gamma\times(0,T)} << ||u||_{E\times(0,T)} \sim e^{-c_0 T}$:

Large boundary inputs and T yield contradiction!

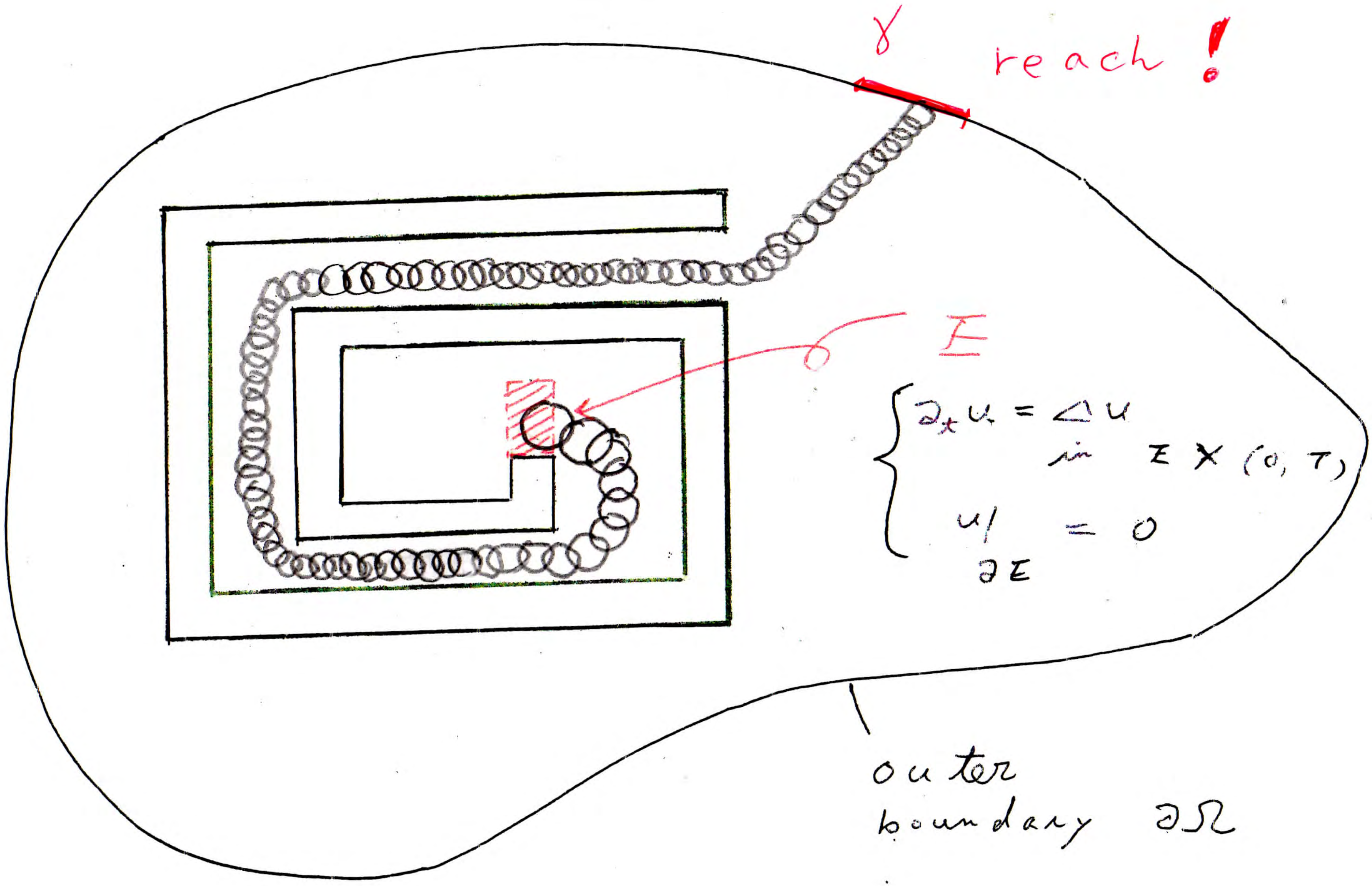




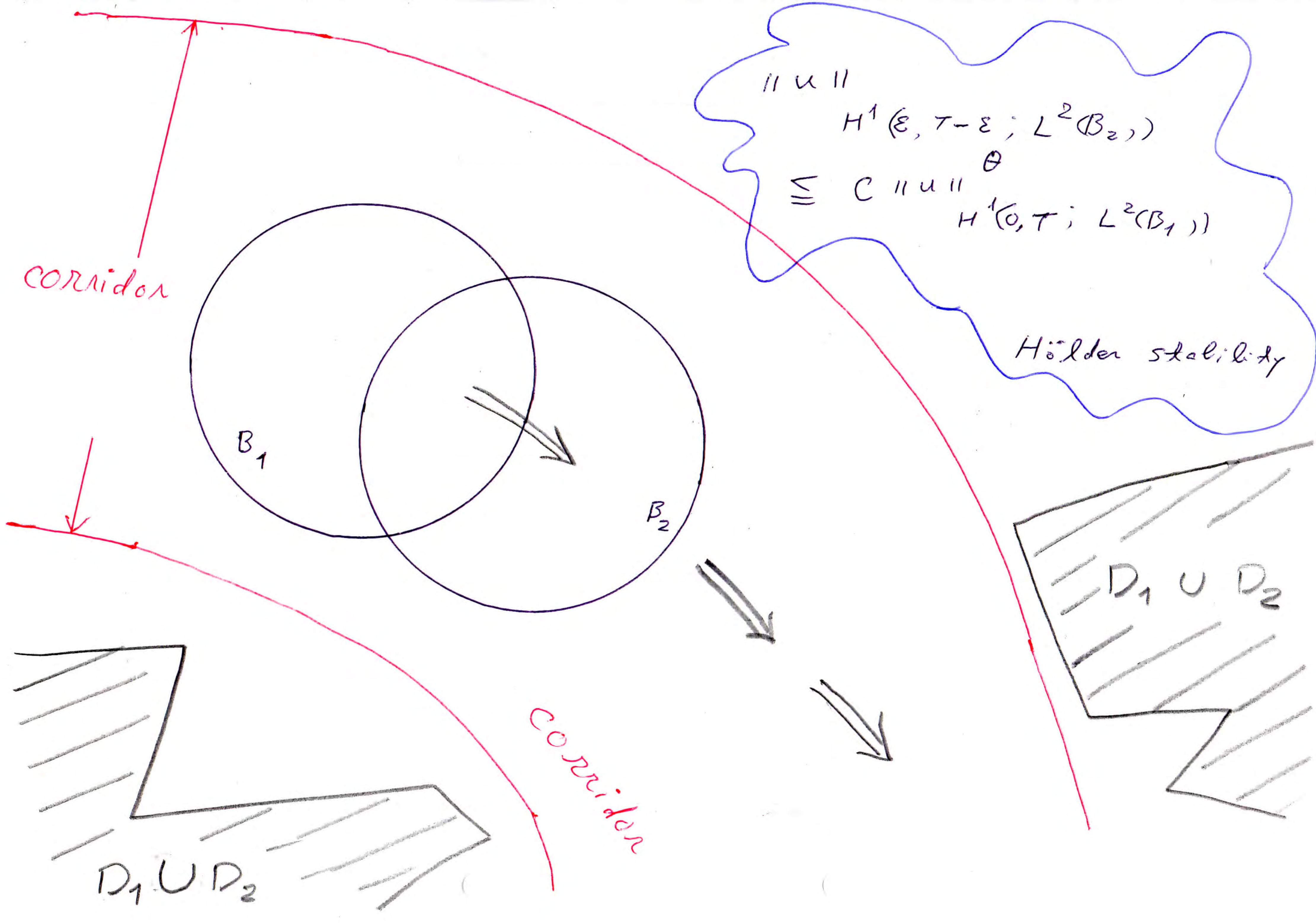
complicated geometry case



E
$\partial_t u = \Delta u$ in $E \times (0, T)$
$u|_{\partial E} = 0$
outer boundary $\partial\Omega$



γ
reach !
E
$\partial_t u = \Delta u$ in $E \times (0, T)$
$u|_{\partial E} = 0$
outer boundary $\partial\Omega$



$\|u\|_{H^1(\varepsilon, T-\varepsilon; L^2(B_2))} \leq C \|u\|^{\theta}_{H^1(0,T; L^2(B_1))}$
Hölder stability
corridor
corridor
B_1
B_2
$D_1 \cup D_2$
$D_1 \cup D_2$

§3. Uniqueness by Bang-bang inputs

$$\partial_t u = \Delta u + \mu(t) f(x) \quad \text{in } \Omega \setminus \overline{D_1}, \quad u|_{\partial(\Omega \setminus D_1)} = 0$$

and

$$\partial_t v = \Delta v + \mu(t) f(x) \quad \text{in } \Omega \setminus \overline{D_2}, \quad v|_{\partial(\Omega \setminus D_2)} = 0.$$

f: interior input amplitude, supp f is small included in $\Omega \setminus \overline{(D_1 \cup D_2)}$:
for example, supp f is close to $\partial\Omega$,

$$\mu(t) = \begin{cases} 1, & 0 \le t \le t_0, \\ 0, & t_0 < t \le T. \end{cases}$$

Theorem 2 (Uniqueness by Bang-bang iput).
If $\partial_\nu u = \partial_\nu u$ *on* $\gamma \times (0, T)$, *then* $D_1 = D_2$.

Key. *Let $\widetilde{\Omega} \subset \mathbb{R}^n$ be bounded domain,*
$f \not\equiv 0$ in $\widetilde{\Omega}$ and

$$\mu(t) = \begin{cases} 1, & 0 \le t \le t_0, \\ 0, & t_0 < t \le T. \end{cases}$$

and

$$\begin{cases} \partial_t U = \Delta U + \mu(t) f(x) \quad \text{in } \widetilde{\Omega} \times (0,T), \\ U|_{\partial\widetilde{\Omega}} = 0. \end{cases}$$

Then U is not time-analytic in any domain E outside of supp f.

Remarks. (1) E must be open set:
Let $f = \varphi$ be some eigenfunction of $-\Delta$ for λ, and $\varphi(x_0) = 0$. Then
$U(x,t) = \varphi(x) \int_0^t e^{-\lambda(t-s)} \mu(s) ds$ is solution and for any μ,
$U(x_0, t) \equiv 0$ is time analytic.

(2) Conjecture: Let subdomain $E \subset\subset \widetilde{\Omega} \setminus$ supp f. If $U|_E$ is time analytic, then μ is time analytic?
("equivalence" of time-analyticity of data and solution!)

Proof of Key $\Longrightarrow$ Theorem 2.

Assume $D_1 \neq D_2$ and supp f is outside of $D_1, D_2 \Longrightarrow$ Then $\exists E \subset\subset (\Omega \setminus \overline{D_1}) \cap D_2$

$$\begin{cases} \partial_t u = \Delta u & \text{in } E \times (0, T), \\ u|_{\partial E} = 0. \end{cases}$$

$\Longrightarrow$ $u|_E$ is time analytic: $u(t) = e^{-tA} u(0)$ in E

Moreover

$$\begin{cases} \partial_t u = \Delta u + \mu(t) f(x) & \text{in } \Omega \setminus \overline{D_1}, \\ u|_{\partial(\Omega \setminus \overline{D_1})} = 0. \end{cases}$$

The key implies $u|_E$ is not time analytic: contradiction ■

Proof of Key

$$U(t) = e^{-tA}U(0) + \int_0^t (e^{-(t-s)A} f)\mu(s)ds, \quad t > 0.$$

Let χ restriction operator: $\chi v := v|_E$ Then

$$\chi U(t) = \begin{cases} \chi A^{-1} f - \chi e^{-tA} A^{-1} f + \chi e^{-tA} U(0), & 0 < t \le t_0, \\ \chi e^{-(t-t_0)A} A^{-1} f - \chi e^{-tA} A^{-1} f + \chi e^{-tA} U(0), & t_0 < t \le T \end{cases}$$

$\Longrightarrow$

$$H(t) = \begin{cases} \chi A^{-1} f, & 0 < t \le t_0, \\ \chi e^{-(t-t_0)A} A^{-1} f, & t_0 < t \le T \end{cases}$$

is analytic in $t > 0$.

$\Longrightarrow A^{-1} f = 0$ in $E \Longrightarrow f = 0$ in $\widetilde{\Omega}$. $\blacksquare$

Thank you very much for your attention!