On obstacle problem for Brakke's mean curvature flow with Neumann boundary condition

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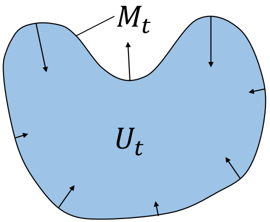
Definition

Let $d \geq 2$ and $U_t \subset \mathbb{R}^d$ be an open set with the smooth boundary M_t, $\forall\, t \in [0, T)$. The family of the hypersurfaces $\{M_t\}_{t\in[0,T)}$ is called a mean curvature flow (MCF) if the following hold:

$$v = h, \quad \text{on } M_t,\ t \in (0, T). \tag{1}$$

v : normal velocity vec. of M_t, h : mean curvature vec. of M_t

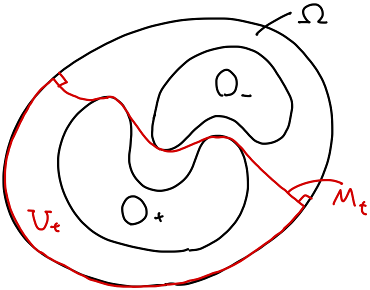


Introduction

Problem

Let $O_+, O_- \subset \mathbb{R}^d$ be open sets (with the smooth boundaries). In this talk, for the mean curvature flow $\{M_t\}_{t\in[0,T)}$ with Neumann boundary condition, we add the following restriction:

$$O_+ \subset U_t \qquad \text{and} \qquad U_t \cap O_- = \emptyset$$



Introduction

Known results (Existence of MCF with obstacles)

- Almeida-Chambolle-Novaga (2012)
  Global existence of flat MCF ($d \geq 2$), and short time existence of viscosity solution ($d = 2$), when $\partial O_\pm : C^{1,1}$
- Mercier-Novaga (2015)
  Short time existence of viscosity solution ($d \geq 2$), and Global existence when $\partial O_\pm$ are graphs.
- Ishii-Kamata-Koike (2017)
  Global existence of the viscosity solution to $\frac{u_t}{|\nabla u|} = \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right)$ with $f \leq u \leq g$, where $f, g \in W^{2,\infty}$.
- T. (2021)
  Global existence of Brakke flow when $d = 2, 3$, $\Omega = \mathbb{T}^d$, and $\partial O_\pm : C^{1,1}$.

Introduction

Theorem (Nik-T., submitted)

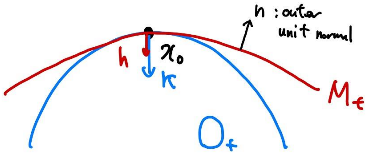
Let $d \geq 2$, $\Omega \subset \mathbb{R}^d$ be a bounded domain with the smooth boundary, $U_0 \subset \Omega$ be open set, $M_0 := \partial U_0 \setminus \partial\Omega$ is smooth, $\partial O_+, \partial O_-$ are $C^{1,1}$. Then there exists a Brakke's mean curvature flow $\{\mu_t\}_{t\in[0,\infty)}$ with generalized Neumann boundary condition and obstacles.

More details will be provided later.

Remark

- We use the phase field method for the proof.
- The Neumann boundary condition for Brakke's MCF is studied by Mizuno-Tonegawa (2015), Kagaya (2019), and Edelen (2020).

Idea of Mercier-Novaga (2015) : For simplicity, let $d = 2$ (i.e. M_t is a curve) and κ be the curvature vector of the obstacle.



Then $|\kappa| \geq |h|$ at x_0. Therefore we consider the MCF with forcing term:

$$v = h + gn, \quad \text{on } M_t,\ t \in (0, \infty),$$

where

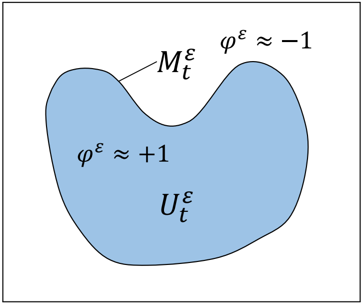
$$g(x) \approx \begin{cases} C, & \text{if } x \in O_+, \\ -C, & \text{if } x \in O_-, \\ 0, & \text{otherwise}, \end{cases} \qquad C := \max_{x \in \partial O_\pm} |\kappa(x)|$$

Phase field method

First we recall the phase field method for MCF.

Let $\varepsilon > 0$, $\Omega := \mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$ and $W(s) := \dfrac{(1-s^2)^2}{2}$. We consider

$$\begin{cases} \varepsilon \varphi^\varepsilon_t = \varepsilon \Delta \varphi^\varepsilon - \dfrac{W'(\varphi^\varepsilon)}{\varepsilon}, & (x,t) \in \Omega \times (0,\infty), \\ \varphi^\varepsilon(x,0) = \varphi^\varepsilon_0(x), & x \in \Omega. \end{cases} \tag{AC}$$



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Remark (Convergence to MCF)

Let $M^\varepsilon_t := \{x \in \Omega : \varphi^\varepsilon(x,t) = 0\}$ and $\{M_t\}_{t \in [0,T)}$ be a smooth MCF. Roughly speaking,

$$M^\varepsilon_0 \to M_0 \quad \Rightarrow \quad M^\varepsilon_t \to M_t \ \ \forall\, t \in [0,T) \ \text{ as } \varepsilon \to 0.$$

(see Bronsard-Kohn (1991), X. Chen (1992), Evans-Soner-Souganidis (1992), Abels-Moser (2019, 2022), Laux-Simon (2018), Moser (2023))

Phase field model

We consider

$$\begin{cases} \varepsilon\varphi^\varepsilon_t = \varepsilon\Delta\varphi^\varepsilon - \dfrac{W'(\varphi^\varepsilon)}{\varepsilon} + g^\varepsilon\sqrt{2W(\varphi^\varepsilon)}, & (x,t)\in\Omega\times(0,\infty), \\ \frac{\partial\varphi^\varepsilon}{\partial\nu}(x,t) = 0, & (x,t)\in\partial\Omega\times(0,\infty), \\ \varphi^\varepsilon(x,0) = \varphi^\varepsilon_0(x), & x\in\Omega, \end{cases} \tag{AC'}$$

where

$$g^\varepsilon(x) \approx \begin{cases} C, & \text{if } x\in O_+, \\ -C, & \text{if } x\in O_-, \\ 0, & \text{otherwise,} \end{cases} \qquad C>0 \text{ (given later)}$$

Remark

$$\sqrt{2W(\varphi^\varepsilon)} = 1-(\varphi^\varepsilon)^2 \approx \begin{cases} 1, & \text{on } M^\varepsilon_t = \{\varphi^\varepsilon(\cdot,t)=0\}, \\ 0, & \text{otherwise.} \end{cases}$$

The forcing term affects only on M^ε_t $\Rightarrow$ (AC') has good property !
Note: T. (2017,2023) used a similar estimate for volume preserving MCF.

Definition of Brakke's MCF

Let $\{M_t\}_{t\geq 0}$ be a $(d-1)$-dim smooth MCF in $\mathbb{R}^d$ $(\partial M_t = \emptyset)$. Then for any non-negative $\phi \in C_c^1(\mathbb{R}^d \times [0,\infty))$ and $0 \leq t_1 < t_2 < \infty$ we have

$$\int_{M_t} \phi \, d\mathcal{H}^{d-1} \Big|_{t=t_1}^{t_2} = \int_{t_1}^{t_2} \int_{M_t} \Big(-\phi|h|^2 + \nabla\phi \cdot h + \frac{\partial \phi}{\partial t}\Big) \, d\mathcal{H}^{d-1} dt.$$

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$$=: \mathcal{B}(M_t, \phi, t_1, t_2) \qquad \text{(Brakke's ineq.)}$$

Proposition

Let $M_t \subset \mathbb{R}^d$ be a smooth hypersurface for $t \geq 0$. Then the following are equivalent:

1. *$\{M_t\}_{t\geq 0}$ is a MCF.*
2. *For any non-negative $\phi \in C_c^1(\mathbb{R}^d \times [0,\infty))$ and $0 \leq t_1 < t_2 < \infty$, M_t satisfies Brakke's ineq.*

Definition of Brakke's MCF with boundary (Characterization of the B.C.)

For any $g \in C^1_c(\mathbb{R}^d;\mathbb{R}^d)$, we consider the first variation of M_t:

$$\delta V_t(g) := \int_{M_t} \mathrm{div}_{M_t} g \, d\mathcal{H}^{d-1} = -\int_{M_t} h \cdot g \, d\mathcal{H}^{d-1} + \int_{\partial M_t} \gamma \cdot g \, d\mathcal{H}^{d-2}.$$

Thus $\delta V_t \lfloor_{\partial\Omega}(g) = \int_{\partial M_t} \gamma \cdot g \, d\mathcal{H}^{d-2}$. Therefore if $\gamma \equiv \nu$ on $\partial M_t \cap \partial\Omega$, then for any $g \in C^1_c(\mathbb{R}^d;\mathbb{R}^d)$,

$$\delta V_t \lfloor^{\top}_{\partial\Omega}(g) := \delta V_t \lfloor_{\partial\Omega}(g - (g\cdot\nu)\nu) = \int_{\partial M_t} \nu \cdot (g - (g\cdot\nu)\nu) \, d\mathcal{H}^{d-2} = 0.$$

Roughly speaking, we define Neumann B.C. for the varifold by

$$\delta V_t \lfloor^{\top}_{\partial\Omega}(g) = 0, \qquad \forall\, g \in C^1_c(\mathbb{R}^d;\mathbb{R}^d).$$

(Mizuno-Tonegawa (2015) proved $\|\delta V_t \lfloor^{\top}_{\partial\Omega} + \delta V_t \lfloor_{\Omega}\| \ll \|V_t\|$ a.e. $t \geq 0$)

Main results

We assume the following:

1. $d \geq 2$, $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary.
2. M_0 is C^1 and $M_0 \perp \partial\Omega$ on $\partial\Omega \cap \partial M_0$.
3. There exists $R_0 > 0$ s.t.

$$O_\pm = \bigcup_{B_{R_0}(x) \subset O_\pm} B_{R_0}(x)$$

and $\operatorname{dist}(O_+, O_-) > 0$, $O_+ \subset U_0$, $\operatorname{dist}(M_0, O_\pm) > 0$.

Remark

$$\mu_t^\varepsilon(\phi) := \frac{1}{\sigma} \int_\Omega \phi \left(\frac{\varepsilon |\nabla \varphi^\varepsilon(x,t)|^2}{2} + \frac{W(\varphi^\varepsilon(x,t))}{\varepsilon} \right) dx, \ \phi \in C_c(\Omega).$$

Here, φ^ε is a sol. of (AC') and $\sigma := \int_{-1}^1 \sqrt{2W(s)}\, ds$.
Under some suitable conditions, $\exists$ $(d-1)$-rectifiable set M_t, $\exists\, \theta_t : M_t \to \mathbb{Z}_{>0}$, $\exists \{\varepsilon_i\}_{i=1}^\infty$ s.t.

$$\mu_t^{\varepsilon_i} \rightharpoonup \mu_t := \theta_t \mathcal{H}^{d-1} \lfloor_{M_t}.$$

Main results

Theorem (Nik-T., submitted)

There exist a families of Radon measures $\{\mu_t\}_{t\in[0,\infty)}$ and Caccioppoli sets $\{U_t\}_{t\in[0,\infty)}$ s.t. the following hold:

1. $\mu_0 = \mathcal{H}^{d-1}\lfloor_{M_0}$ and for a.e. $t \geq 0$, μ_t is integral on $\Omega \setminus \overline{(O_+ \cup O_-)}$, and there exists a generalized mean curvature vector $h(\cdot, t) \in L^2(\mu_t)$ in the sense of Mizuno-Tonegawa.
2. For any $t \geq 0$, $\partial^* U_t \subset \operatorname{spt} \mu_t$, $O_+ \subset U_t$, $U_t \cap O_- = \emptyset$, $\operatorname{spt} \mu_t \cap O_\pm = \emptyset$.
3. $\forall$ non-negative $\phi \in C^1(\overline{\Omega} \times [0,\infty))$ with $\operatorname{spt} \phi(\cdot, t) \cap O_\pm = \emptyset$ and $\nabla \phi(\cdot, t) \cdot \nu = 0$ on $\partial\Omega$ for $\forall\, t \geq 0$, and $0 \leq \forall\, t_1 < \forall\, t_2 < \infty$,
$$\int_\Omega \phi \, d\mu_t \Big|_{t=t_1}^{t_2} \leq \int_{t_1}^{t_2} \int_\Omega \{(-h\phi + \nabla\phi) \cdot h + \phi_t\} \, d\mu_t dt.$$
4. μ_t satisfies Neumann B.C. in the sense of Mizuno-Tonegawa.

Sub solution and super solution

Assume that $B_{R_0}(0) \subset O_+$ (the assumption "$\partial O_\pm$ is $C^{1,1}$" comes from $R_0 > 0$). For $x \in B_{R_0+\delta}(0)$, we set

$$\underline{r}(x) := -\frac{c_1}{(R_0+\delta)^2 - |x|^2} + c_2,$$

$$c_1 := \frac{\left((R_0+\delta)^2 - R_0^2\right)^2}{2R_0}, \qquad c_2 := \frac{(R_0+\delta)^2 - R_0^2}{2R_0},$$

$$\underline{u}^\varepsilon(x) := \tanh(\underline{r}(x)/\varepsilon) \qquad \text{and} \qquad \overline{u}^\varepsilon(x) := \tanh(-\underline{r}(x)/\varepsilon).$$

Lemma

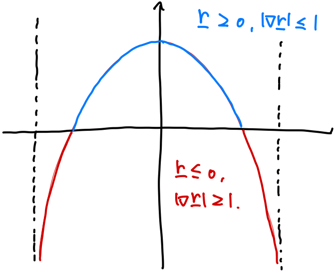
For sufficiently large $C > 0$ (Recall: $C = \|g^\varepsilon\|_{L^\infty}$), $\underline{u}^\varepsilon$ is a sub solution to (AC') if $B_{R_0}(0) \subset O_+$.

Sub solution and super solution

Set r^ε by $\varphi^\varepsilon = \tanh(r^\varepsilon/\varepsilon)$. Then,

$$\begin{aligned} \varphi^\varepsilon_t &= \Delta\varphi^\varepsilon - \frac{W'(\varphi^\varepsilon)}{\varepsilon^2} + g^\varepsilon\sqrt{2W(\varphi^\varepsilon)} \\ \Rightarrow \quad r^\varepsilon_t &= \Delta r^\varepsilon + \frac{W'(\varphi^\varepsilon)}{\varepsilon\sqrt{2W(\varphi^\varepsilon)}}(|\nabla r^\varepsilon|^2 - 1) + g^\varepsilon \\ &= \Delta r^\varepsilon + \frac{2\tanh(r^\varepsilon/\varepsilon)}{\varepsilon}(1 - |\nabla r^\varepsilon|^2) + g^\varepsilon. \qquad \text{(R)} \end{aligned}$$

Hence we only need to check that $\underline{r}$ is the sub solution to (R).



Thank you for your kind attention !