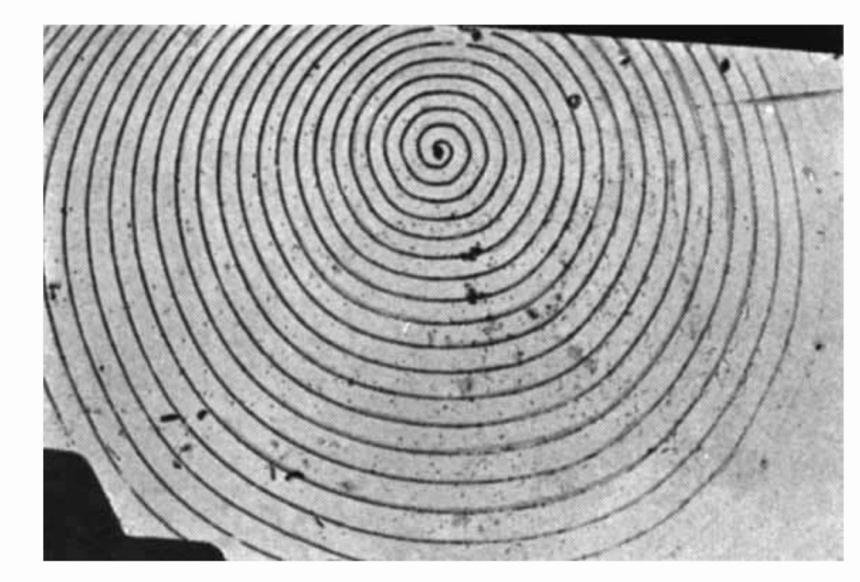
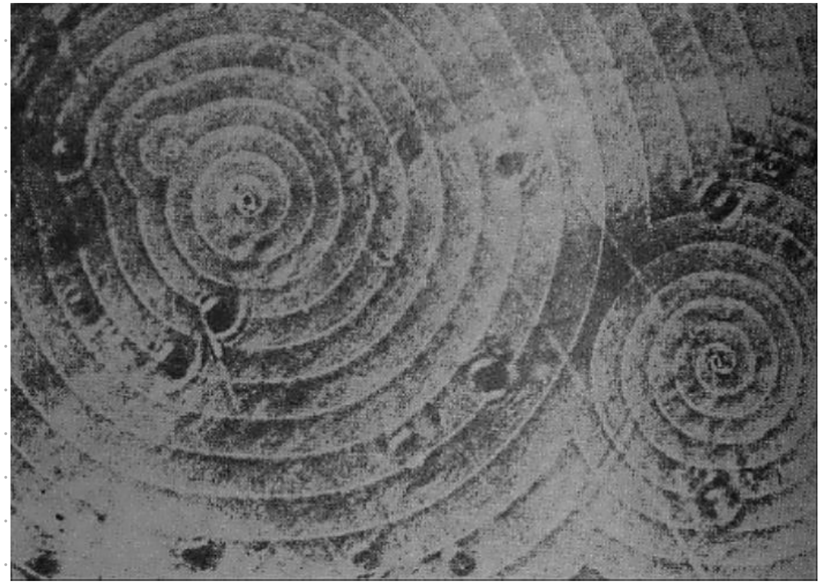
On asymptotic growth rate of solutions to level-set forced mean curvature flows with evolving spirals

Hiroyoshi MITAKE (U. Tokyo) Joint work with H. V. Tran (UW-Madison)

§ Introduction (Spiral crystal growth)



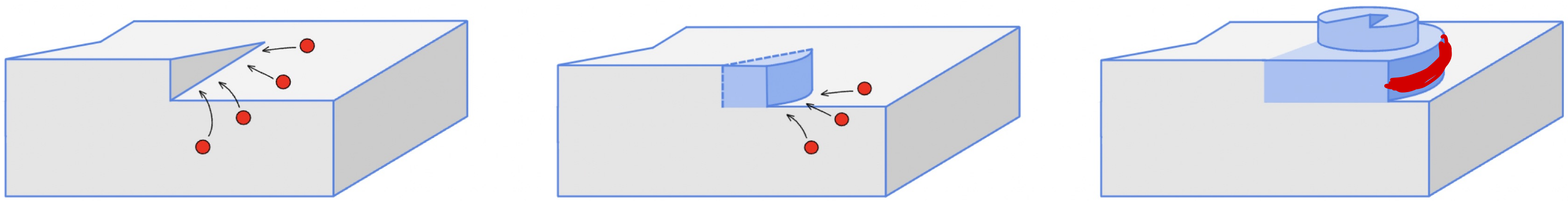


S. Amelinckx (1951, Nature)

Spiral appears in

crystal growth, Belousov-Zhabotinsky reaction etc.

Crystal Growth by screw dislocation



Burton-Cabrera-Frank (1951) established a theory of crystal growth.

The front moves according to $V = v_\infty (\rho_c \kappa + 1)$ on Γ_t.

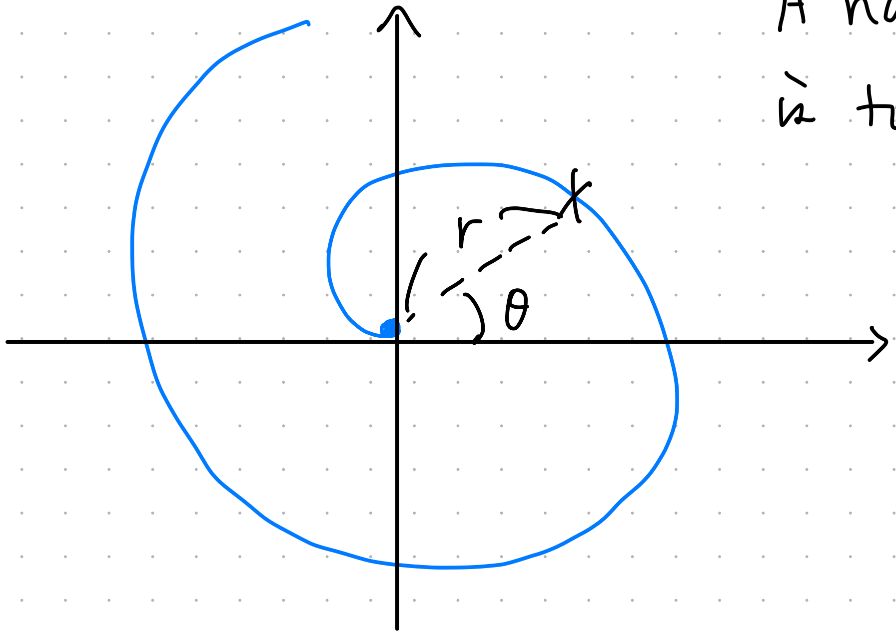
Here, V : outward normal velocity,

Γ_t : a curve in $\mathbb{R}^2$

κ : curvature

given $\begin{cases} v_\infty > 0 : \text{velocity of straight line steps} \\ \rho_c > 0 : \text{critical radius for the generation of two dimensional kernel from supersatulation.} \end{cases}$

View from above



A naive way to describe this behavior is to use the polar coordinate

$$\Gamma_t = \left\{ (r\cos\theta, r\sin\theta) \,\middle|\, r \geq 0,\ \theta \in \mathbb{R},\ u(r,t) - \theta \equiv 0 \mod 2\pi\mathbb{Z} \right\}$$

for some function $u : [0,\infty) \times [0,\infty) \to \mathbb{R}$.

$$\Leftrightarrow\ r u_t = \frac{r u_{rr}}{1 + r^2 u_r^2} + u_r \left(\frac{2 + r^2 u_r^2}{1 + r^2 u_r^2} \right) + c\sqrt{1 + r^2 u_r^2} \quad \text{for } (r,t) \in (0,\infty) \times (0,\infty) \qquad (*)$$

Ref. Giga - Ishimura - Kohsaka (2002)

Forcadel - Imbert - Monneau (2012, 2015)

A phase-field approach : Karma - Plapp (1998)

Ogiwara - Nakamura (2003)

If there are multiple spirals, the polar coordinate never work!

Level set approach based on the theory of viscosity solutions.

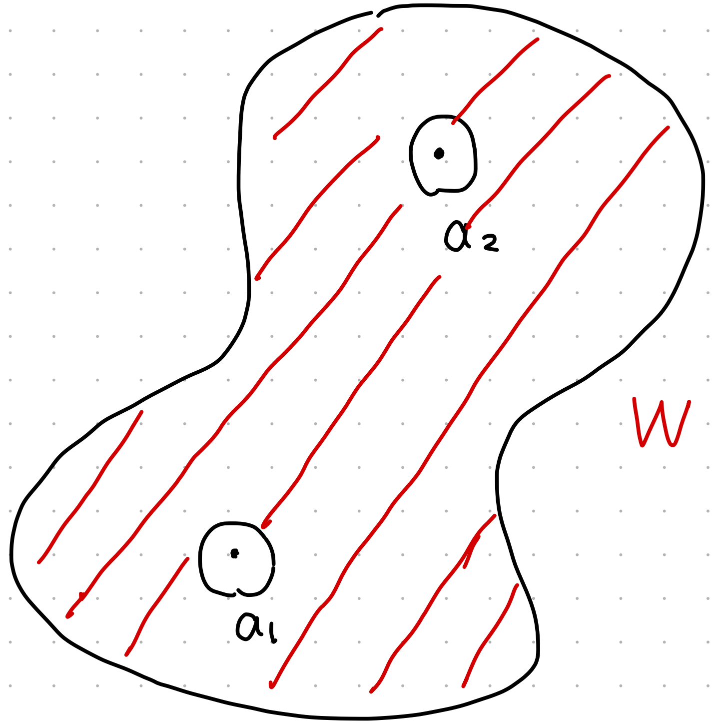
Setting

Let $\Omega \subset \mathbb{R}^2$ be a bounded C^2 domain,

$$W := \Omega \setminus \bigcup_{j=1}^{N} B(a_j, r)$$

for $a_j \in \Omega$, $r > 0$ (small) with

$$\overline{B(a_i, r)} \cap \overline{B(a_j, r)} = \emptyset \text{ if } i \neq j.$$



Consider a sheet structure function

$$\theta(x) = \sum_{j=1}^{N} m_j \arg(x - a_j)$$

multi-valued function

where $m_j \in \mathbb{Z} \setminus \{0\}$ are given.

Note that $D\theta$ is single-valued. Indeed,

$$D\theta(x) = \sum_{j=1}^{N} \frac{m_j\left(-(x_2 - a_{j,1}),\ x_1 - a_{j,2}\right)}{|x - a_j|^2}.$$

Now, let's consider

$$(N)\begin{cases} u_t = \left(\operatorname{div}\left(\dfrac{D(u-\theta)}{|D(u-\theta)|}\right) + c(x)\right)|D(u-\theta)| & \text{in } W \times (0,\infty) \\ D(u-\theta)\cdot n = 0 & \text{on } \partial W \times (0,\infty) \\ u(\cdot,0) = u_0 & \text{on } \overline{W} \end{cases}$$

where

$$(A1)\begin{cases} u_0 \in C^2(\overline{W}) \text{ with } D(u_0-\theta)\cdot n = 0, \\ c \in C^1(\overline{W}) \text{ with } c(x) > 0 \quad \forall x \in \overline{W}. \end{cases}$$

Known Results.

Ohtsuka (2003), Goto-Nakagawa-Ohtsuka (2008) establish the well-posedness (Existence / Comparison principle).

Numerical results by Ohtsuka-Tsai-Giga (2015, 2018).

Smerka (2000)

P. Smereka / Physica D 138 (2000) 282–301

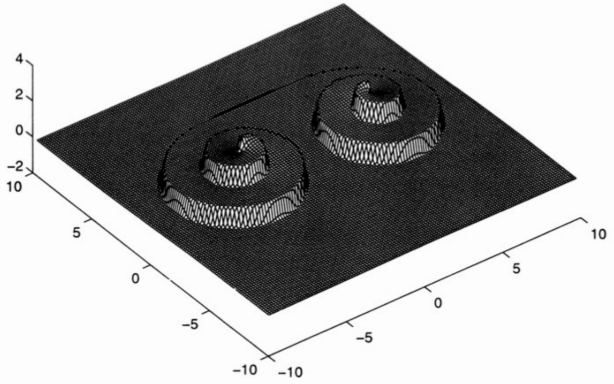


Fig. 5. This shows the height profile for the corresponding step-line shown in Fig. 4 at $t = 5$.

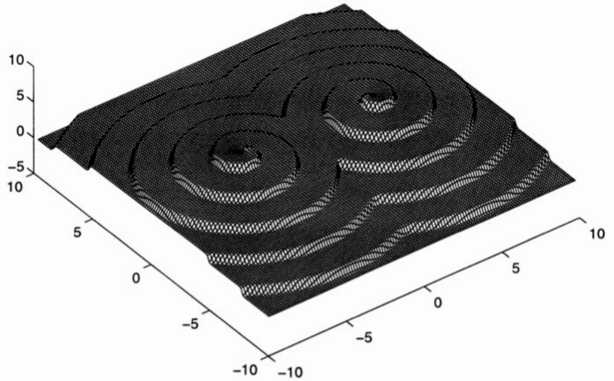


Fig. 6. This shows the height profile for the corresponding step-line shown in Fig. 4 at $t = 11$.

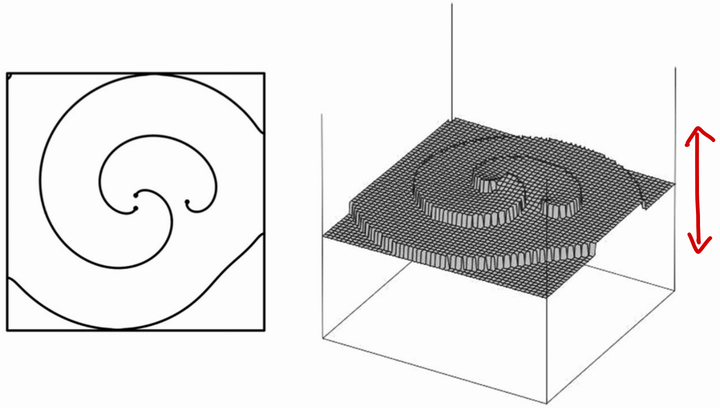


Figure 2. Example of level set for spirals and its height function.

$h(x,t) \approx u(x,t)$

1918

DOI: 10.1021/acs.cgd.7b00833
Cryst. Growth Des. 2018, 18, 1917–1929

Our interest is a behavior of the height function h.

Definition of h. Let

$$\Theta(x) := \sum_{j=1}^{N} m_j \, \Theta_j(x),$$

where Θ_j is the principal value of $\arg(x - a_j)$,

take $k(x,t) \in \mathbb{Z}$ so that

$$-\pi \leq u(x,t) - \left(\Theta(x) + 2\pi k(x,t)\right) < \pi.$$

For $h_0 > 0$ (the unit height of step), define

$$h(x,t) := \frac{h_0}{2\pi}\left\{\Theta(x) + 2\pi k(x,t) + \pi \,\mathrm{sign}\left(u(x,t) - \left(\Theta(x) + 2\pi k(x,t)\right)\right)\right\}.$$

$\Longrightarrow$ $\boxed{h(x,t) \text{ "}\asymp\text{" } u(x,t)}$

Our goal. Want to understand

(a) Existence of the asymptotic speed, that is,

$$\exists \lim_{t \to +\infty} \frac{h(x,t)}{t} \quad ?$$

(b) Qualitative / Quantitative properties of the asymptotic speed.

§ Main Results

$$(N)\begin{cases} u_t = \left(\operatorname{div}\frac{D(u-\theta)}{|D(u-\theta)|} + c(x)\right)|D(u-\theta)| \\ D(u-\theta)\cdot n = 0 \\ u(\cdot,0) = u_0 . \end{cases}$$

Thm 1. Assume (A1). There exists a constant $S_c \in [0,\infty)$ s.t.

$$\lim_{t\to+\infty} \frac{\max_{x\in\overline{W}} u(x,t)}{t} = S_c .$$

$\Rightarrow$ To show the existence of $\lim_{t\to+\infty}\frac{u(x,t)}{t}$,

it is important to the time-global Lipschitz estimate of u.

Thm 2. Assume (A1) holds, and

(A2) $\exists\, \delta > 0$ s.t.

$$c(x)^2 - 2|Dc(x)| - 2C_0|c(x)| - \frac{8C_0}{K_0} \geqslant 0$$

where $C_0 := \max\left\{ C_1, \frac{1}{r} \right\}$,

$C_1 := \max\left\{ -\lambda \mid \lambda \text{ is a curvature at } x_0 \in \partial\Omega \right\}$.

Then $\exists\, L = L(\|u_0\|_{C^2}, \|D\theta\|_{C^1}, \|c\|_{C^1}, C_0, K_0, \delta) > 0$

s.t. $\|u_t\|_{L^\infty(W\times[0,\infty))} + \|Du\|_{L^\infty(W\times[0,\infty))} \leq L$.

Rem. If (A2) does NOT hold, $\exists$ example to show that u is not time global Lip. conti.

Thm 3. Assume (A1), (A2) hold, and let $S_c \in [0,\infty)$ be the constant given by Thm 1. Then,

$$\frac{u(x,t)}{t} \to S_c \text{ uniformly for } x \in \overline{W} \text{ as } t \to +\infty.$$

Moreover,

$$\frac{h(x,t)}{t} \to \frac{h_0 S_c}{2\pi} \text{ uniformly for } x \in \overline{W} \text{ as } t \to +\infty.$$

Comment. S_c is an important object to be studied more deeply.

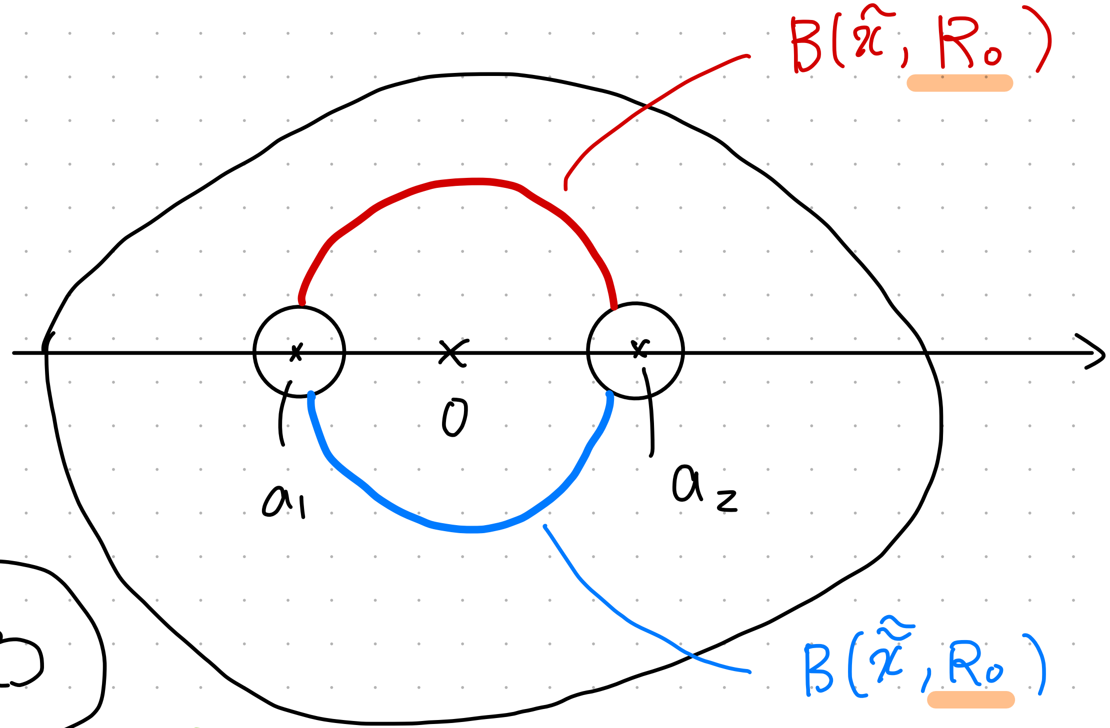
Example. Assume $N = 2$, $a_1 = (l, 0)$, $a_2 = (-l, 0)$

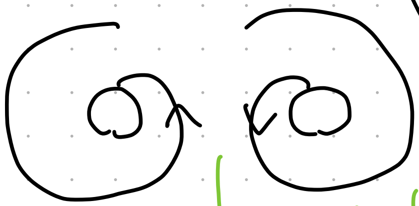
for some

$0 < r < l \leq R_0 := \dfrac{1}{\max_{\overline{W}} c}$

and $m_1 = -m_2$.

$\Rightarrow$ Inactive pair





$\rightarrow$ interfere (干渉)

Assume that

Ω is a domain like the picture.

Then, u is bounded. In particular, $S_c = 0$.

Remark ∘ When $C(x) \equiv c \in (0,\infty)$, this result is announced by Ohtsuka (2013, Oberwolfach report).

∘ Burton – Cabrera – Frank (1951)

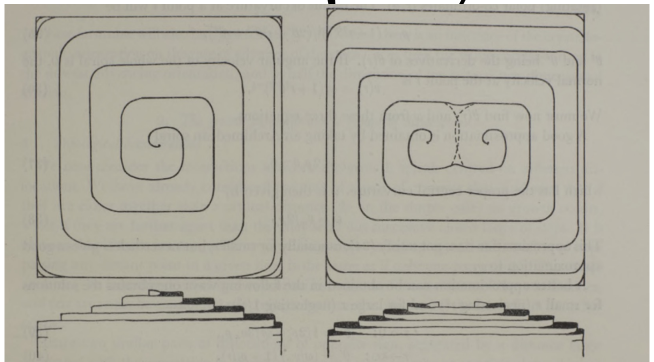


FIGURE 3. Growth pyramid due to a single screw dislocation.

FIGURE 4. Growth pyramid due to a pair of dislocations.

9·1. *Topological considerations*

We now consider the interactions between the growth spirals centred on different dislocations. We have already considered the case of a pair of opposite sign, and seen that if they are closer together than a critical distance ($2\rho_c$ in the simple case) no growth occurs, while if they are further apart than this they send out successive closed loops of steps. It is obvious that if there are *two such pairs* these loops unite on meeting, and the number of steps

→ We can expect $\delta_c > 0$ if l is large enough, and

This is still OPEN (Numerically, see Ohtsuka – Tsai – Giga (2015))

Also, BCF gives an interesting observation.

- Assume $N = 2$, $a_1 = (\ell, 0)$, $a_2 = (-\ell, 0)$,

  $m_1 = m_2 = 1$.

In this setting, intuitively it seems that two spirals accelate. However, even to prove $\delta_c > 0$ is hard (OPEN).

$\Longrightarrow$ Find a good barrier function (subsol) from below.

Hard !

Thm 4. Assume (A1), (A2) hold, and $\delta_c = 0$.

Then, $\exists\, v \in \mathrm{Lip}(\overline{\Omega})$ s.t. $u(\cdot,t) \to v$ in $C(\overline{W})$ as $t \to +\infty$,

where

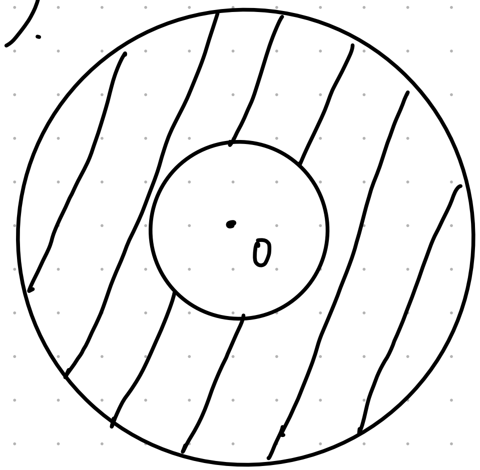
$$\begin{cases} -\left(\operatorname{div}\left(\dfrac{D(v-\theta)}{|D(v-\theta)|}\right) + c(x)\right)|D(v-\theta)| = 0 & \text{in } W \\ D(v-\theta)\cdot n = 0 & \text{on } \partial W \end{cases}$$

Rem. When $\delta_c > 0$, the large time behavior for (N) is rather open.

Example (Special Sol)

$N=1,\ m_1=1,\ a_1=0,\ W= B(0,R)\setminus B(0,r),\ c(x)=|x|.$

Assume $u_0(x)=g\left(\frac{x}{|x|}\right)$ for some $g\in C^2(\mathbb{R}^2)$.



Then, $u(x,t)= g\left(R_{-t}\frac{x}{|x|}\right)+t$

for $\forall (x,t)\in \overline{W}\times[0,\infty)$,

where

$$R_r = \begin{pmatrix} \cos r & -\sin r \\ \sin r & \cos r \end{pmatrix}.$$

<u>Rem.</u> Note that $u(x,t)-t$ does NOT converge as $t\to+\infty$.

This is a new type of special solution associated with (N) as far as I know.

Thanks for your attention!