A SIMPLE AND EFFECTIVE PDE MODEL FOR BUSHFIRES

THEORY AND COMPUTATION

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Motivation

Wildfires are a global phenomenon. In Australia they have been especially impactful. Let me give a brief impression of what a massive problem they are for Australia.

The remains of a burnt property in Bruthen, SA.

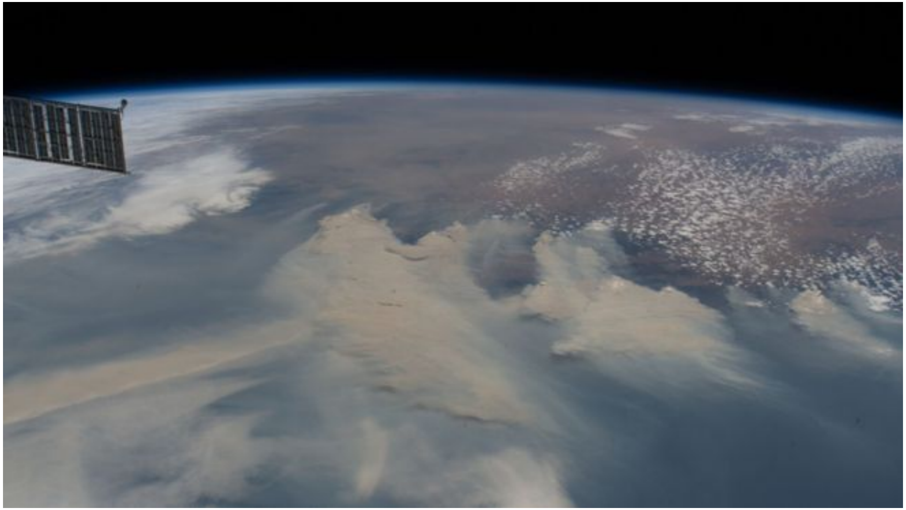
Motivation

Local air quality.

Melbourne, Australia, pre-COVID.

MOTIVATION

Dangerous smoke crosses borders.



Bushfire smoke seen from space crossing the Pacific.

Motivation

Mental health cost.

A showground being transformed into an evacuation centre.

MOTIVATION

Economic cost.

Destroyed buildings in Eden-Monaro.

Motivation

Climate feedback.

Jetbourne photograph of smoke plumes from East Gippsland.

MOTIVATION

Habitat damage.

Ash and debris on Boydtown beach.

Motivation

Agricultural.

Sheep moving through fire grounds in Bega.

MOTIVATION

An out of control fire line in Bowraville, NSW.

Emergency response to bushfires, especially at the catastrophic rating level, needs to be (drastically) more efficient. Our small contribution is to enhance the existing modelling and prediction technology.

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The "ultimate" universal bushfire model is not available yet, and it is customary to approach the problem under different perspectives, so that often the different models are classified into the categories of "physical" (or "theoretical"), "semi-physical" (or "semi-empirical"), and "empirical", see e.g. Perry '98, Pastor-Zárate-Planas-Arnaldos '03; sometimes "mathematical" models have been considered as a separate category closely related to "simulation", see e.g. Sullivan '09 x3.

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- qualitative understanding of the main processes involved
- fast simulations.

We feel as though this would make the model reliable and concretely usable in real-time.

Simple models can also act as initial blueprints from which more complex models can be built.

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More specifically, the equation on which we focus our attention has the form

$$\partial_t u = c\Delta u + \int_{\mathbb{R}^n} \big(u(y,t) - \Theta(y,t)\big)_+ K(x,y)\,dy + \left(\left(\omega - \frac{\beta(u)\nabla u}{|\nabla u|^\alpha}\right)\cdot \nabla u\right)_- , \tag{1}$$

where u, $\Theta : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ and $c : \mathbb{R}^n \times \mathbb{R} \to (0,+\infty)$ are real functions depending on the space point $x \in \mathbb{R}^n$ and time $t \in \mathbb{R}$, $K : \mathbb{R}^n \times \mathbb{R}^n \to [0,+\infty)$ is an interaction kernel, $\omega : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a time-dependent vector field, $\beta : \mathbb{R} \to \mathbb{R}$, $\alpha \in [0,2]$, and we used the notation for the "positive and negative parts" of a function

$$f_+(x) := \max\{0, f(x)\} \text{ and } f_-(x) := \max\{0, -f(x)\}. \tag{2}$$

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Interpretation. In this model, $u = u(x,t)$ represents the temperature of the environment at the space point $x \in \mathbb{R}^n$, at time t (in many applications, one can focus on the case $n = 2$, and deal with a given region with suitable, e.g. Dirichlet, conditions).

Diffusion, combustion, wind, and convection

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As customary, one assumes that temperature spreads through a **heat equation**, in which $c = c(x,t)$ denotes the environmental diffusion coefficient (the case of constant c modeling a homogeneous environment).

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The **combustion** is described by an effective ignition temperature $\Theta = \Theta(x,t)$, above which the fuel burns. The case of a constant $\Theta(x,t) = \Theta_0$ represents the ideal situation of fuel homogeneously spread across the environment, with infinite abundance of fuel at every point. In this case, the fuel burns at temperature Θ_0 and affects nearby points through the interaction potential K. Note that when the temperature u of the fuel is below the effective ignition temperature Θ, the combustion effect disappears, in view of the positive part of the function considered in the integral).

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Situations in which the availability of fuel varies in space and time are of course modeled by nonconstant functions Θ. We stress that the model can also include a memory effect, in which the burned fuel turns into ash and is not any longer subject to combustion. This can be done, for instance, by considering the case in which

$$\Theta(x,t) := \begin{cases} \tan\left(\frac{\pi}{2F}\int_0^t \big(u(x,\tau) - \bar{\Theta}(x)\big)_+ d\tau\right) & \text{if } \int_0^t \big(u(x,\tau) - \bar{\Theta}(x)\big)_+ d\tau < F, \\ +\infty & \text{otherwise.} \end{cases} \tag{3}$$

In this framework, $F = F(x,t)$ accounts for the fuel available (for instance, F can be taken to be constant; the case $F = F(x)$ corresponds to the situation of an initial amount of fuel $F(x)$ available at the point x, and the case

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Moreover, the constitutional ignition temperature at a given point x is $\bar{\Theta}(x)$ (the case of a constant ignition temperature $\bar{\Theta}(x) = \Theta_0$ accounting for a uniformly distributed fuel). Above this ignition temperature, the fuel burns, and the amount of fuel burned is described by the quantity

$$B(x,t) := \int_0^t \big(u(x,\tau) - \bar{\Theta}(x)\big)_+ \, d\tau. \tag{4}$$

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Indeed, comparing (3) and (4), the effective ignition temperature in this case can be written in the form

$$\Theta = \tan\left(\frac{\pi B}{2F}\right),$$

hence when the quantity of burned combustible B reaches the threshold F, the ignition term in (1) vanishes.

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The effect of the environmental **wind** is modeled by a vector $\omega = \omega(x,t)$ inducing a transport term in equation (1).

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The **convection** is encoded by an additional wind term of the form $\frac{\beta(u)\nabla u}{|\nabla u|^\alpha}$. The function β can be thought as supported above the ignition temperature, so that this term is active only in the burning region, directed along, or opposite to (depending on the sign of β) the gradient temperature. The exponent α can be taken in the interval $[0,2]$ to further modulate the intensity of this term with respect to the temperature gradient.

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The negative part in the term regarding the environmental wind and the convective term accounts for the fact that wind can spread the fire faster, but cannot stop it, hence these terms provide a contribution only when their direction is oriented opposite to the gradient temperature, which is in turn a proxy for the direction of the front.

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We stress that the total wind term, consisting of the superposition of environmental and pyrogenic winds, provides a contribution only when this function takes negative values. This means that the wind and convective terms provide a contribution only when their direction is oriented opposite to the gradient temperature, that is in the direction of the fire front (which goes from higher to lower values of the temperature, and note that, in our notation (2), the negative part of a function is nonnegative).

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In the absence of environmental wind, the pyrogenic flow takes the form $\frac{(-\beta)_-}{|\nabla u|^{\alpha-2}} = \frac{\beta_+}{|\nabla u|^{\alpha-2}}$, contributing only for positive values of β. In this spirit, the role of the negative part in the last term of (1) is to *spread* the fire in the direction of the wind: as the simulations confirm, without the negative part, the burning region will tend to mainly *translate* in the direction of the wind.

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Terms related to a fire-induced wind (also known as “fire wind” or “pyrogenic flow”) have been introduced in the literature to account for the significant buoyant upflow over the fire region, created by the strong temperature gradient, producing local low pressure, which acts as a sink with horizontal pressure gradient and draws in the surrounding air (see Smith-Morton-Leslie ’75).

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The possibility of modeling the wind advection by a transport term, in various forms, has also been previously considered in the literature, see e.g. equation (2.2) in Harris-McDonald '22. In terms of concrete applications, the combination of pyrogenic and environmental wind can be a delicate issue: indeed, on the one hand, in case of wind-driven fires it is expected the effect of pyrogenic wind to be less prominent (see Beer '91), on the other hand pyrogenic winds can be produced even by relatively small wildfires (as observed in Lareau-Clements '17).

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As a sanity check, we remark that, since both the positive and negative parts of a function are, in our notation (2), nonnegative, the solutions of (1) are always *supersolutions of the heat equation* (which is consistent, from the physical point of view, with the fact that bushfires can only increase the environmental temperature).

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We have begun the development of existence theory for this model, for which we present the following two theorems.

Local and Global Well-posedness

Theorem 1 (DVWW '24)

Let Ω be a bounded domain of $\mathbb{R}^n$ with C^2 boundary. Let also $K \in L^2(\Omega \times \Omega)$, $\Theta \in L^\infty(\Omega \times (0,1))$, $\omega \in L^\infty(\Omega \times (0,1))$, $\beta \in W^{1,\infty}(\mathbb{R})$, and $\gamma \in (1,2]$.
Then, there exists $T_\star \in (0,1]$, depending only on n, Ω, $\|K\|_{L^2(\Omega\times\Omega)}$, $\|\omega\|_{L^\infty(\Omega\times(0,1))}$, and $\|\beta\|_{W^{1,\infty}(\mathbb{R})}$, such that if $T \in (0, T_\star]$ then, given $g \in H_0^1(\Omega)$, the problem

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possesses a solution $u \in L^2((0,T), H_0^1(\Omega))$.

Theorem 2 (DVWW '24, convective terms with linear scaling)

Let Ω be a bounded domain of $\mathbb{R}^n$ with C^2 boundary. Let also $K \in L^2(\Omega \times \Omega)$, $\Theta \in L^\infty(\Omega \times (0,1))$, $\omega \in L^\infty(\Omega \times (0,1))$ and $\beta \in W^{1,\infty}(\mathbb{R})$.
Then, there exist $\varepsilon_0 \in (0,1)$, depending only on n, Ω and $\|\Theta_-\|_{L^\infty(\Omega\times(0,1))}$, and $T_\star \in (0,1]$, depending only on n, Ω, $\|K\|_{L^2(\Omega\times\Omega)}$, $\|\omega\|_{L^\infty(\Omega\times(0,1))}$, and $\|\beta\|_{W^{1,\infty}(\mathbb{R})}$, such that if

$$\|K\|_{L^2(\Omega\times\Omega)} + \|\omega\|_{L^\infty(\Omega\times(0,1))} + \|\beta\|_{L^\infty(\mathbb{R})} \leqslant \varepsilon_0$$

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Local and Global Well-posedness

Theorem 3 (DVWW '24)

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Numerical validation and real-time simulations

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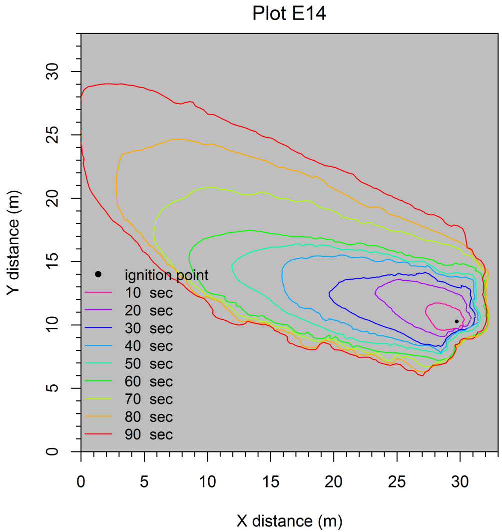
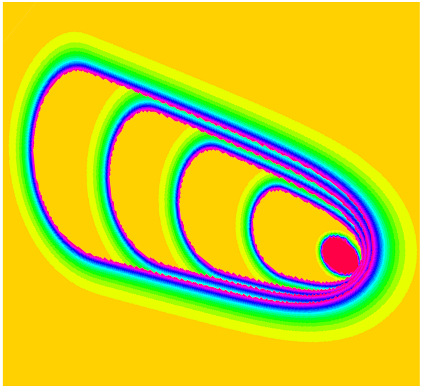
We remark that we did not make special effort to change parameters in (1) drastically to fit pictures, but instead tried to make only natural, broad choices. In a real-world scenario, we expect that parameters are at best approximated, so fine-tuning them would be counterproductive in our view.

Numerical validation and real-time simulations





Observed isochrones for spot fire E14 investigated in Cruz-Sullivan-Kidnie-Hurley-Nichols '16, compared to isochrones from our simulation taken at evenly spaced iterations of the scheme. The red line is the fire front with temperature above ignition, and temperature decreasing below ignition as the colours progress from blue to green, then yellow.

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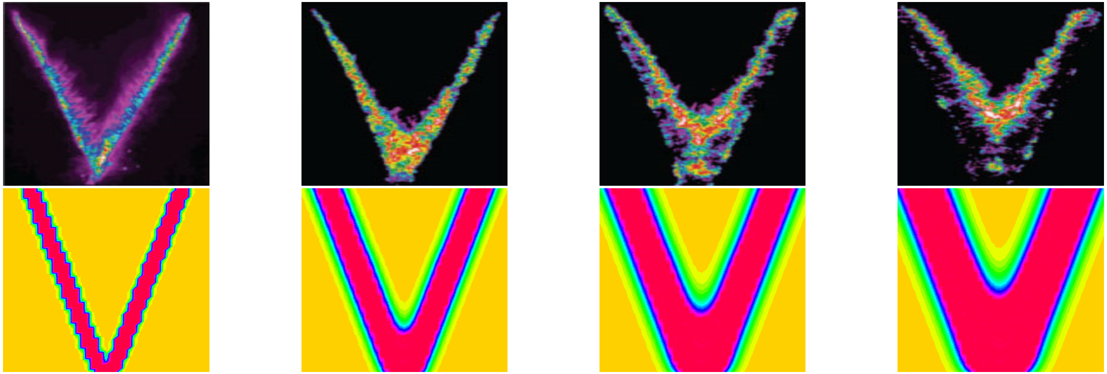

Experiment CF-14 from Viegas-Raposo-Davim-Rossa '12, with a 30 degree cone angle. Frames taken from an infrared camera at thirty second intervals. Simulation frames taken at evenly spaced iterations of the numerical simulation.

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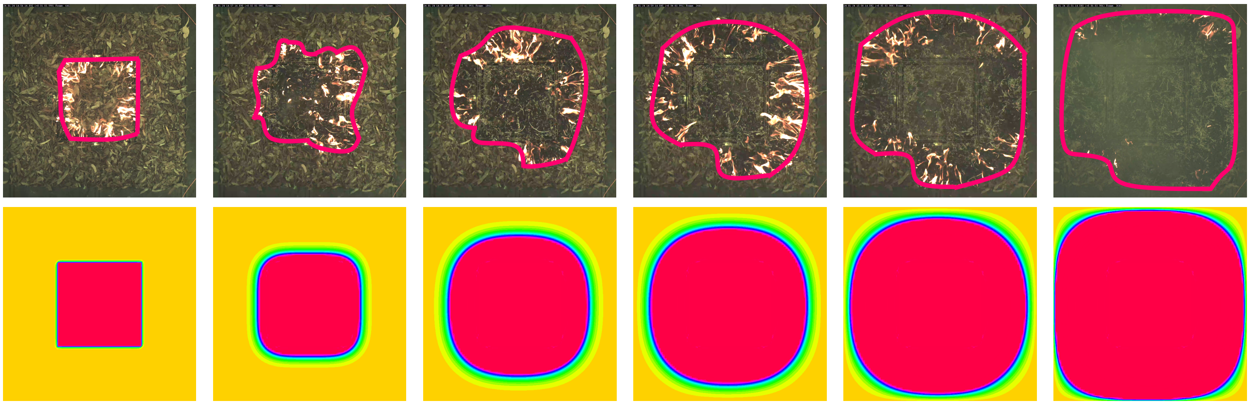

Experimental vs. Simulation Frames. Experiment conducted at the Pyrotron CSIRO Canberra February 2024. Experimental frames taken at one-minute intervals. A magenta outline is traced around the approximate fire line. Simulation frames taken at evenly spaced iterations of the numerical simulation.

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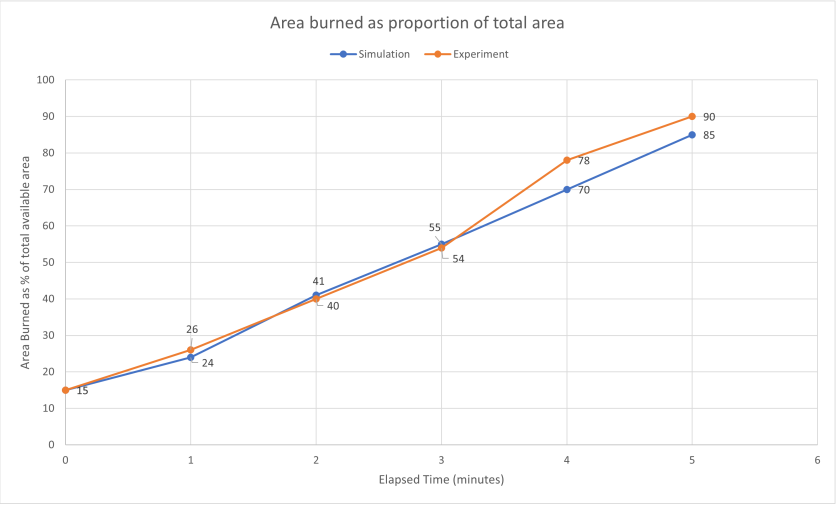

Here we take the approximated burned area as a measure of validation. The matching is almost perfect after up to 3 minutes, and a small deviation takes place at minute 4 and minute 5. This small deviation has also to be expected, since our simulations are based on ideal Dirichlet conditions, while the experimental fire may be affected by slightly approximated boundary conditions allowing for a faster diffusion near the boundary. Note also the coarseness of the fuel used for the experimental fire.

CONCLUSIONS AND THE HYBRID GEOMETRIC PROBLEM

We have put forth a model for bushfires relying on partial differential equations:

$$\partial_t u = c\Delta u + \int_{\mathbb{R}^n} \big(u(y,t) - \Theta(y,t)\big)_+ K(x,y)\,dy + \left(\left(\omega - \frac{\beta(u)\nabla u}{|\nabla u|^{\alpha}}\right)\cdot \nabla u\right)_- .$$

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The front of the fire can be modeled by the level set of the temperature function corresponding to the ignition temperature. The velocity of the front in the normal direction when the ignition temperature Θ_0 is constant takes the form

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- The simulations run in real-time and are in agreement both with the data collected in supervised lab experiments and with those coming from real bushfire events.

That's all folks!

Thanks for your attention!

THAT'S ALL FOLKS! I

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