PDEs on evolving domains and evolving finite elements

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Mathematical Aspects for Interfaces and Free Boundaries

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Idea

Develop unified methodology
within applied and numerical analysis suitable for the
simulation of complex interface and free boundary motion

- Functional analytic framework for abstract PDEs
- Time dependent function spaces
- Link domain evolution to evolution equation on domains
- Approximate time dependent space by evolving finite element spaces
- Evolving bulk and surface domains approximated by fitted triangulated domains
- Avoid unfitted finite elements and level set equations
- In this talk focus on evolving surfaces

Colleagues and Others

- Dziuk, Deckelnick
- Ranner, Alphonse, Stinner, Venkataraman
- Church, Djurdjevac, Kornhuber
- Hatcher, Caetano, Grassellli,Poiatti
- Sales,Mavrakis
- Garcke, Kovacs
- Reusken+... Miura.
- Apologies to the many who are missing

For each $t \in [0,T]$, let $\Gamma(t) \subset \mathbb{R}^{n+1}$ be a compact (i.e., bounded and no boundary) n-dimensional hypersurface of class C^2, and assume the existence of a flow $\Phi\colon [0,T] \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ such that for all $t \in [0,T]$, with $\Gamma_0 := \Gamma(0)$, the map $\Phi_t^0(\cdot) := \Phi(t,\cdot)\colon \Gamma_0 \to \Gamma(t)$ is a C^2-diffeomorphism that satisfies

$$\begin{aligned} \frac{d}{dt}\Phi_t^0(\cdot) &= \mathbf{w}(t,\Phi_t^0(\cdot)) \\ \Phi_0^0(\cdot) &= \mathrm{Id}(\cdot). \end{aligned} \tag{1}$$

We think of the map $\mathbf{w}\colon [0,T] \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ as a velocity field, and we assume that it is sufficiently smooth and in particular in C^2. A normal vector field on the hypersurfaces is denoted by $\nu\colon [0,T] \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$.
The normal velocity is $\mathbf{w}_\nu = \mathbf{w}_\nu \cdot \nu\nu$.
A bulk domain $\Omega(t)$ with boundary $\Gamma(t)$ may be viewed as sub manifold in $\mathbb{R}^{n+2}$.

Parameterised evolving surfaces

Associated with the map Φ_t^0 is a parameterisation .
Let $\Gamma(t) \subset \mathbb{R}^3$ be a closed surface parametrised by X over an initial surface Γ^0:

$$\Gamma(t) = \{X(p,t) \,:\, p \in \Gamma^0\}.$$

Surface velocity $\mathbf{w}$ satisfies

$$\partial_t X(p,t) = \mathbf{w}(X(p,t),t).$$

The canonical parameterisation would be the normal flow for which the velocity field is in the normal direction.

Normal time derivative Suppose that the velocity field associated to the evolving hypersurface $\{\Gamma(t)\}$ is $\mathbf{w} = \mathbf{w}_\nu + \mathbf{w}_\tau$ where $\mathbf{w}_\nu$ is the normal velocity field and $\mathbf{w}_\tau$ is a tangential velocity field. In this case, the formula

$$\partial^\circ u = u_t + \nabla u \cdot \mathbf{w}_\nu$$

defines the *normal time derivative* $\partial^\circ u$.

For our purposes the **material derivative** is associated with the parameterisation of the hypersurface and depends on the tangential velocity.

$$\partial^\bullet u = \partial^\circ u + \mathbf{w}_\tau \cdot \nabla_\Gamma u$$

A **physical material derivative** would be

$$\dot{u} = \partial^\bullet u + (\mathbf{v}_\tau - \mathbf{w}_\tau) \cdot \nabla_\Gamma u$$

where $\mathbf{v}_\tau$ is a tangential physical material velocity.

Choosing w_τ for some purpose of computation or analysis may be appropriate. In numerical methods this is called the *Arbitrary Lagrangian Eulerian (ALE)* approach where it is employed to yield *good meshes*.

Differential operators on Γ

- Outward normal vector: ν
- Tangential gradient: $\nabla_\Gamma u == \nabla\bar{u} - (\nabla\bar{u}\cdot\nu)\nu : \Gamma \to \mathbb{R}^3$
- Laplace–Beltrami operator: $\Delta_\Gamma u = \nabla_\Gamma \cdot \nabla_\Gamma u$
- extended Weingarten map (3×3 symmetric matrix)

$$A(x) = \nabla_\Gamma \nu(x)$$

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$$\text{mean curvature} \qquad H = \mathrm{tr}(A) = \kappa_1 + \kappa_2,$$
$$\text{and} \qquad |A|^2 = \|A\|_F^2 = \kappa_1^2 + \kappa_2^2.$$

We will write PDEs in a way that is independent of the parametrisation of the domains i.e. point wise in space time on the evolving domains.
Using a version of Reynolds transport formula for evolving domains we may derive PDEs on evolving domains via

- Balance laws
- Gradient flow
- Variational

Advection-diffusion on an evolving surface

Let $\Gamma(t)$ be a time (t) dependent $n-$dimensional hypersurface in $\mathbb{R}^{n+1}$.

$$\partial^{\circ}\mathrm{u} + \nabla_{\Gamma} \cdot (\mathscr{B}_{\Gamma}\mathrm{u}) - \nabla_{\Gamma} \cdot (\mathcal{A}_{\Gamma} \nabla \mathrm{u}) + \mathscr{C}_{\Gamma}\mathrm{u} = 0 \qquad \text{on } \Gamma(t)$$

$$\mathrm{u}(\cdot, 0) = \mathrm{u}_0 \qquad \text{on } \Gamma_0 := \Gamma(0)$$

$\mathcal{A}_{\Gamma}$ is a smooth diffusion tensor which maps the tangent space of Γ into itself,
$\mathscr{B}_{\Gamma}$ is a tangential vector field,
$\mathscr{C}_{\Gamma}$ is a smooth scalar field.
$\partial^{\circ}\mathrm{u}$ denotes the normal time derivative
i.e. the time derivative of a function along a trajectory on $\Gamma(t) \times t$ moving in the direction normal to $\Gamma(t)$.

Let $\Gamma(t) = \partial\Omega(t)$where $\Omega(t)$ is a time dependent bulk domain in $\mathbb{R}^{n+1}$.

$$\mathrm{u}_t + \nabla \cdot (\mathscr{B}_\Omega \mathrm{u}) - \nabla \cdot (\mathcal{A}_\Omega \nabla \mathrm{u}) + \mathscr{C}_\Omega \mathrm{u} = 0 \qquad \text{on } \Omega(t)$$

$$(\mathcal{A}_\Omega \nabla \mathrm{u} - \mathscr{B}_\Omega \mathrm{u}) \cdot \nu + \alpha \mathrm{u} - \beta \mathrm{v} = 0 \qquad \text{on } \Gamma(t)$$

$$\mathrm{u}(\cdot, 0) = \mathrm{u}_0 \qquad \text{on } \Omega_0 := \Omega(0)$$

$$\partial^\circ \mathrm{v} + \nabla_\Gamma \cdot (\mathscr{B}_\Gamma \mathrm{v}) - \nabla_\Gamma (\mathcal{A}_\Gamma \mathrm{v}) + \mathscr{C}_\Gamma \mathrm{v} + (\mathcal{A}_\Omega \nabla \mathrm{u} - \mathscr{B}_\Omega \mathrm{u}) = 0 \qquad \text{on } \Gamma(t)$$

$$\mathrm{v}(\cdot, 0) = \mathrm{v}_0 \qquad \text{on } \Gamma_0 := \Gamma(0)$$

where α and β are positive constants.

Let $\Gamma(t)$ be a time (t) dependent $2-$dimensional hypersurface in $\mathbb{R}^3$.
Seek a triple (u, p_1, p_2) to the problem:

$$u \cdot \nu_\Gamma = V_\Gamma, \quad (p_2) \qquad \text{on } \cup_{t\in I} \{t\} \times \Gamma(t)$$

$$\partial^\circ u + u \cdot \nabla_\Gamma u + \nabla_\Gamma p_1 + 2\mu_0 \nabla_\Gamma \cdot E(u) = -p_2 \nu + f \qquad \text{on } \cup_{t\in I} \{t\} \times \Gamma(t)$$

$$\nabla_\Gamma \cdot u = 0 \quad (p_1) \qquad \text{on } \cup_{t\in I} \{t\} \times \Gamma(t)$$

$$E_\Gamma(\mathrm{v}) = \frac{\nabla_\Gamma \mathrm{v} + (\nabla_\Gamma \mathrm{v})^T}{2}, .$$

Note: two Lagrange multipliers, (p_1, p_2).

Geometric gradient flow

Concentration dependent energy

$$\mathcal{E}(\Gamma, u) = \int_\Gamma G(u),$$

The (L^2, H^{-1})-gradient flow of $\mathcal{E}$ yields the *coupled geometric flow*:

$$\begin{aligned} \mathrm{v} &= -g(u) H \nu_\Gamma = V \nu_\Gamma, \\ \partial^\bullet u + uVH &= \Delta_\Gamma G'(u), \end{aligned}$$

with $g(u) = G(u) - uG'(u)$.

Definition (Bochner-type spaces)

Define the spaces

$$L^2_X = \{u : [0,T] \to \bigcup_{t\in[0,T]} X(t) \times \{t\}, t \mapsto (\bar{u}(t),t) \mid \phi_{-(\cdot)}\bar{u}(\cdot) \in L^2(0,T;X_0)\}$$
$$L^2_{X^*} = \{f : [0,T] \to \bigcup_{t\in[0,T]} X^*(t) \times \{t\}, t \mapsto (\bar{f}(t),t) \mid \phi^*_{(\cdot)}\bar{f}(\cdot) \in L^2(0,T;X^*_0)\}.$$

More precisely, these spaces consist of equivalence classes of functions agreeing almost everywhere in $[0,T]$, just like ordinary Bochner spaces.

For $u \in L^2_X$, we will make an abuse of notation and identify $u(t) = (\bar{u}(t),t)$ with $\bar{u}(t)$ (and likewise for $f \in L^2_{X^*}$).

Theorem

The spaces L^2_X and $L^2_{X^*}$ are Hilbert spaces with the inner products

$$\begin{aligned}(u,v)_{L^2_X} &= \int_0^T (u(t),v(t))_{X(t)}\,\mathrm{d}t \\ (f,g)_{L^2_{X^*}} &= \int_0^T (f(t),g(t))_{X^*(t)}\,\mathrm{d}t.\end{aligned} \tag{2}$$

Abstract strong and weak material derivatives

Definition (Strong material derivative)

For $\xi \in C^1_X$ define the strong material derivative $\dot{\xi} \in C^0_X$ by

$$\dot{\xi}(t) := \phi_t \left(\frac{d}{dt} (\phi_{-t} \xi(t)) \right).$$

- We see that the space C^1_X is the space of functions with a strong material derivative, justifying the notation.
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Definition (Weak material derivative)

For $u \in L^2_{\mathcal{V}}$, if there exists a function $g \in L^2_{\mathcal{V}^*}$ such that

$$\int_0^T \langle g(t), \eta(t) \rangle_{\mathcal{V}^*(t), \mathcal{V}(t)} = -\int_0^T (u(t), \dot{\eta}(t))_{\mathcal{H}(t)} - \int_0^T \lambda(t; u(t), \eta(t))$$

holds for all $\eta \in \mathcal{D}_{\mathcal{V}}(0,T)$, then we say that g is the weak material derivative of u, and we write $\dot{u} = g$ or $\partial^\bullet u = g$.

The form λ is identified using the push forward map. This concept of a weak material derivative is indeed well-defined: if it exists, it is unique, and every strong material derivative is also a weak material derivative.

Theorem (Transport theorem and formula of partial integration)

For all u, $v \in W(\mathcal{V},\mathcal{V}^*)$, the map

$$t \mapsto (u(t),v(t))_{\mathcal{H}(t)}$$

is absolutely continuous on $[0,T]$ and

$$\frac{d}{dt}(u(t),v(t))_{\mathcal{H}(t)} = \langle \partial^\bullet u(t),v(t)\rangle_{\mathcal{V}^*,\mathcal{V}(t)} + \langle \partial^\bullet v(t),u(t)\rangle_{\mathcal{V}^*(t),\mathcal{V}(t)} + \lambda(t;u(t),v(t))$$

for almost every $t \in [0,T]$. For all u, $v \in W(\mathcal{V},\mathcal{V}^*)$, the following formula of partial integration holds

$$\begin{aligned}(u(T),v(T))_{\mathcal{H}(T)} &- (u(0),v(0))_{\mathcal{H}_0}\\ &= \int_0^T \langle \partial^\bullet u(t),v(t)\rangle_{\mathcal{V}^*(t)\mathcal{V}(t)} + \langle \partial^\bullet v(t),u(t)\rangle_{\mathcal{V}^*(t)\mathcal{V}(t)}\\ &+ \lambda(t;u(t),v(t))\,\mathrm{d}t.\end{aligned}$$

Model abstract problem

Abstract problem

$$\text{Find } \; u(t) \in \mathcal{V}(t)$$
$$u(0) = u_0 \in \mathcal{V}(0)$$
$$\partial^{\bullet} u + \mathcal{A}(t) u = f \quad \in \mathcal{V}^*(t)$$

written in a variational form as

$$\langle \partial^{\bullet} u, v \rangle_{\mathcal{V}^*(t),\mathcal{V}(t)} + a(t; u, v) = \langle f, v \rangle_{\mathcal{V}^*(t),\mathcal{V}(t)}$$
$$u(0) = u_0$$

with associated (arbitrary) family of Hilbert triples

$$\mathcal{V}(t) \subset \mathcal{H}(t) \subset \mathcal{V}^*(t), \; t \in [0, T]$$

parametrised by $t \in [0, T]$.

Model abstract problem: Saddle point problem

For $t \in \mathbb{R}_+$ let $Y(t)$ and $X(t)$ be, respectively, given families of evolving Hilbert and Banach spaces. We denote the dual $X(t)$ as $X^*(t)$ and assume we have the *Gelfand triple* structure:

$$X(t) \subset Y(t) \subset X^*(t).$$

where we refer to $Y(t)$ as the pivot space. Let $Z(t)$ be an evolving Banach family. We are concerned with the *linear saddle-point* problem:

$$\begin{aligned} \partial^\bullet u(t) + A(t)u(t) + B^*(t)p(t) = f(t) \quad &\in X^*(t), \\ B(t)u(t) = g(t) \quad &\in Z^*(t), \\ u(0) = u_0 \in Y(0). \end{aligned}$$

with $\partial_t^\bullet u$ denoting the material derivative and we seek a pair of solutions (u, p).

ABC Methodology

- Construct finite dimensional spaces as analogues of the continuous spaces
- Approximation theory
- Construct discrete analogues of bilinear forms in variational setting
- Well posedness of discrete problem
- Perturbation bounds for bilinear forms
- Error analysis via well posedness of continous problem and consistency

PDE and Finite Element setting

PDE analysis

- Domain and function spaces
- PDE: Initial value problem
- Bilinear forms and transport formulae
- Variational formulation
- Verify assumptions

Numerical analysis

- Evolving bulk finite element spaces
- Lifted bulk finite element spaces
- Evolving surface finite element spaces
- Lifted surface finite element spaces
- Discrete material derivatives and transport formulae

All these require precise definitions.

Tasks for realisation of abstract theory

Define

- Evolving finite element
- Evolving triangulation
- Evolving finite element space

Establish

- Approximation properties
- Lifted evolving spaces

Realise

- $\Omega_h(t)$ and $\Gamma_h(t)$ by interpolation, for example.
  Evolving nodes on initial triangulations by velocity field
- $\mathcal{S}_h(t)$

Establish

- Discrete bilinear forms
- Approximation estimates
- Ritz projection and for material derivative

Surface finite elements

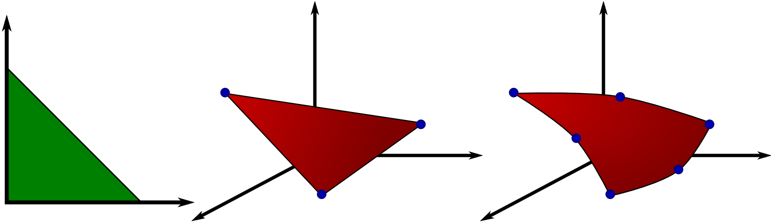

Figure: Examples of different surface finite elements in the case $n = 2$. Left shows a reference finite element (in green), centre shows an affine finite element and right shows an isoparametric surface finite element with a quadratic F_K. The plot shows the element domains in red and the location of nodes in blue.

Evolving isoparametric surface finite element

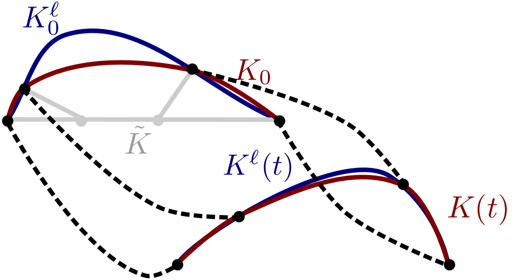

Figure: Examples of construction of an isoparametric evolving surface finite element for $k = 3$. The Lagrange nodes $a_i(t)$ follow the dashed black trajectories from the initial element $K_0 \subset \Gamma_{h,0}$ to a element $K(t) \subset \Gamma_h(t)$.

Model abstract lifted discrete problem

Abstract lifted problem

$$
\begin{aligned}
&\text{Find } \; u_h^\ell(t) \in \mathcal{V}_h^\ell(t)\\
&u_h^\ell(0) = u_0^{h,\ell} \in \mathcal{V}_h^\ell(0)\\
&\partial_h^{\bullet,\ell} u_h^\ell + \mathcal{A}_h^\ell(t) u_h^\ell = f_h^\ell
\end{aligned}
$$

written in a variational form as

$$
\begin{aligned}
\langle \partial_h^{\bullet,\ell} u_h^\ell, \mathrm{v}\rangle_{\mathcal{V}^*(t),\mathcal{V}(t)} + a_h^\ell(t; u_h, \mathrm{v}) &= \langle f_h^\ell, \mathrm{v}\rangle_{\mathcal{V}^*(t),\mathcal{V}(t)}, \forall \mathrm{v} \in \mathcal{V}_h^\ell(t)\\
u_h^\ell(0) &= u_0^{h,\ell}
\end{aligned}
$$

$$\mathcal{V}_h^\ell(t) \subset \mathcal{V}(t)$$

The relationships between evolving function spaces

- $\mathcal{H}(t)$ pivot space
- $\mathcal{V}(t)$ solution spaces
- $\mathcal{Z}_0(t)$ regularity space for dual problem
- $\mathcal{Z}(t)$ higher regularity space for solution with specific data

$$
\begin{tikzcd}
\mathcal{H}(t) \arrow[ddd, leftrightarrow] & \mathcal{V}(t) \arrow[l, hook] \arrow[d, phantom, "\cup" description] \arrow[ddd, leftrightarrow, bend left] & \mathcal{Z}_0(t) \arrow[l, hook] & \mathcal{Z}(t) \arrow[l, hook] \\
& \mathcal{S}_h^{\ell}(t) \arrow[d, leftrightarrow] & & \\
& \mathcal{S}_h(t) \arrow[d, phantom, "\cap" description] & & \\
\mathcal{H}_h(t) & \mathcal{V}_h(t) \arrow[l, hook] & &
\end{tikzcd}
$$

$\subset$ denotes subspace inclusion

$\hookrightarrow$ denotes continuous embedding

$\leftrightarrow$ denotes that the lift is a bijection between these spaces.

- Since $\mathcal{S}_h(t)$ is a closed subspace of $\mathbb{V}_h(t)$ it is a Hilbert space and forms a compatible pair $(\mathcal{S}_h(t), \phi_t^h|_{\mathcal{S}_{h,0}})_{t\in[0,T]}$.
- Well defined spaces $L^2_{\mathcal{S}_h}$ and $C^1_{\mathcal{S}_h}$ and the material derivative $\partial_h^\bullet \chi_h$ is well defined for $\chi_h \in C^1_{\mathcal{S}_h}$.
- Defines the spaces $L^2_{\mathcal{H}_h}, L^2_{\mathcal{V}_h}$ and $C^1_{\mathcal{H}_h}, C^1_{\mathcal{V}_h}$. For $\eta_h \in C^1_{\mathcal{H}_h}$, we denote by $\partial_h^\bullet \eta_h$ the (strong) material derivative) with respect to the push-forward map ϕ_t^h defined by

$$\partial_h^\bullet \eta_h := \phi_t^h\left(\frac{d}{dt}\phi_{-t}^h \eta_h\right).$$

Basis functions and transport property

Let $\{\chi_i(\cdot,0)\}_{i=1}^N$ be a basis of $\mathcal{S}_{h,0}$ and push-forward to construct a time dependent basis $\{\chi_i(\cdot,t)\}_{i=1}^N$ of $\mathcal{S}_h(t)$ by

$$\chi_i(\cdot,t) = \phi_t^h(\chi_i(\cdot,0)).$$

It follows that

$$\partial_h^\bullet \chi_i = 0$$

so that for a decomposition

$$\chi_h(t) := \sum_{i=1}^N \gamma_i(t)\chi_i(t) \qquad \text{for all } \chi_h \in \mathcal{S}_h(t),$$

we compute that

$$\partial_h^\bullet \chi_h = \sum_{i=1}^N \dot{\gamma}_i(t)\chi_i(t) \qquad \text{for all } \chi_h \in C^1_{\mathcal{S}_h}.$$

Another discrete material derivative appropriate for analysis

$\partial_\ell^\bullet \eta$ denotes the material derivative for the push-forward map ϕ_t^ℓ.

$$\partial_\ell^\bullet \eta := \phi_t^\ell \frac{d}{dt}(\phi_{-t}^\ell \eta) \qquad \text{for all } \eta \in C^1_{(\mathcal{H},\phi^\ell)}.$$

This is a different material derivative to the material derivative defined with respect to the push-forward map ϕ_t^h.

Important observation of Dziuk and Elliott, the following commutation result holds:

$$\partial_\ell^\bullet(\eta_h^\ell) = (\partial_h^\bullet \eta_h)^\ell \qquad \text{for all } \eta_h \in C^1_{\mathcal{H}_h}.$$

Indeed:

$$\partial_\ell^\bullet(\eta_h^\ell) = \phi_t^\ell \frac{\mathrm{d}}{\mathrm{d}t}(\phi_{-t}^\ell(\eta_h^\ell)) = \left(\phi_t^h\Big(\big(\frac{\mathrm{d}}{\mathrm{d}t}(\phi_{-t}^h \eta_h)^\ell\big)^{-\ell}\Big)\right)^\ell = \left(\phi_t^h\Big(\frac{\mathrm{d}}{\mathrm{d}t}(\phi_{-t}^h \eta_h)\Big)\right)^\ell = (\partial_h^\bullet \eta_h)^\ell,$$

since the lift at time $t = 0$ and time derivative commute and $(\cdot)^\ell$ and $(\cdot)^{-\ell}$ are inverses.

Lemma

$\eta_h \in C^1_{\mathcal{H}_h}$ if, and only if, $\eta_h^\ell \in C^1_{(\mathcal{H},\phi^\ell)}$, and $\eta_h \in C^1_{\mathcal{V}_h}$ if, and only if, $\eta_h^\ell \in C^1_{(\mathcal{V},\phi^\ell)}$.

Weak and variational formulation of the advection diffusion equation

For every $\varphi(\cdot,t) \in H^1(\Gamma(t))$

Weak form

$$\int_{\Gamma(t)} \partial^\bullet u\varphi + \int_{\Gamma(t)} u\varphi \nabla_\Gamma \cdot v + \int_{\Gamma(t)} \nabla_\Gamma u \cdot \nabla_\Gamma \varphi = 0$$

Variational form

$$\frac{d}{dt}\int_{\Gamma(t)} u\varphi + \int_{\Gamma(t)} \nabla_\Gamma u \cdot \nabla_\Gamma \varphi = \int_{\Gamma(t)} u\partial^\bullet \varphi$$

Abstract variational form

$$\frac{d}{dt}(u,\varphi) + a(u,\varphi) = m(u,\partial^\bullet \varphi).$$

Finite element method

$$\frac{d}{dt}(U_h,\phi_h)_h + a_h(U_h,\phi_h) = (U_h,\partial_h^\bullet \phi_h)_h, \quad U_h(\cdot,0) = U_{h0}. \tag{3}$$

Evolving mass matrix

$$M(t)_{jk} = \int_{\Gamma_h(t)} \chi_j \chi_k,$$

Evolving stiffness matrix

$$\mathcal{S}(t)_{jk} = \int_{\Gamma_h(t)} \nabla_{\Gamma_h}\chi_j \nabla_{\Gamma_h}\chi_k,$$

$U_h = \sum_{j=1}^N \alpha_j \chi_j$, $\alpha = (\alpha_1,\ldots,\alpha_N)$

Algebraic form

$$\frac{d}{dt}(M(t)\alpha) + \mathcal{S}(t)\alpha = 0, \tag{4}$$

which does not explicitly involve the velocity of the surface.

Error analysis: Not quite accurate but near!!

$$\frac{1}{2}\frac{d}{dt}(\theta,\theta) - h + a_h(\theta,\theta) = F_h(\theta).$$

Theorem

Let u be a sufficiently smooth solution satisfying

$$\sup_{t\in(0,T)} \|u\|^2_{H^{k+1}(\Gamma(t))} + \int_0^T \|\partial^\bullet u\|^2_{H^{k+1}(\Gamma(t))} dt < \infty$$

and let $u_h^\ell(,t) = U_h^l(\cdot,t), t \in [0,T]$ be the spatially discrete solution with initial data $u_{h0} = U_{h0}^l$ satisfying

$$\|u(\cdot,0) - u_{h0}^\ell\|_{L^2(\Gamma(0))} \leq ch^{k+1}.$$

Then the error estimate

$$\sup_{t\in(0,T)} \|u(\cdot,t) - u_h^\ell(\cdot,t)\|_{L^2(\Gamma(t))} \leq ch^{k+1}$$

holds for a constant c independent of h.

Outlook

Extensions

- Nonlinear equations
  Evolving Surface Navier-Stokes Cahn-Hilliard system
- Moving boundaries on moving boundaries
  Stefan problems, Geometric Curve Motion on evolving surfaces
- Coupling of bulk surface problems in prescribed evolving domains
  Advection diffusion on bulk and interface domains
- Coupling PDE equations to flow of function spaces
  MCF and diffusion, Flows in moving domains
- Flow maps ϕ allowing good discrete flows
  Harmonic map heat flow, De Turck trick, BGN approach
- Viscosity Solutions PDEs on prescribed evolving surfaces
  Hamilton-Jacobi