Moving Boundary Problems on Moving Cell Boundaries

Asymptotic limits of models for receptor-ligand interactions on moving domains

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a talk based on joint work with

A. Alphonse (WIAS), D. Caetano and C. Elliott (Warwick)

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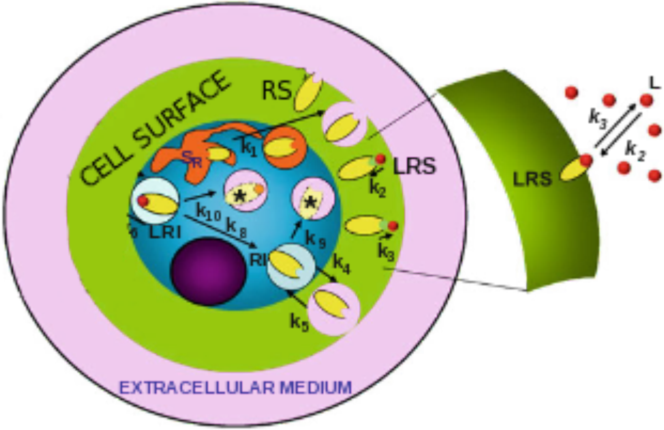
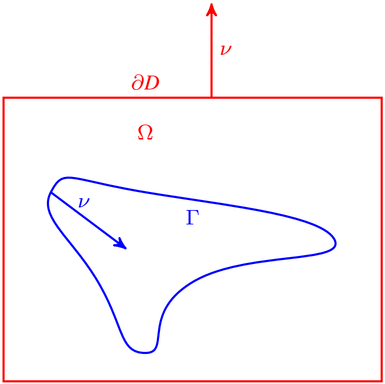
Receptor-ligand dynamics

- Primary mechanism by which cells sense and respond to their environment.
- Important in cell biological phenomena such as proliferation, motility, the maintenance of structure or form, adhesion, cellular signalling, etc.



(García-Peñarrubia, Gálvez, and Gálvez, 2013)

Coupled bulk-surface model for receptor-ligand dynamics (Elliott, Ranner, and Venkataraman, 2017)



- u:Bulk ligand concentration
- w: Surface receptor density
- z: Surface receptor-ligand complex density

Model

$$
\begin{aligned}
\delta_\Omega \partial_t u - \Delta u &= 0 && \text{in } \Omega \\
u = u_D \text{ or } \nabla u \cdot \nu &= 0 && \text{on } \partial D \\
\nabla u \cdot \nu &= \frac{1}{\delta_{k'}} z - \frac{1}{\delta_k} uw && \text{on } \Gamma \\
\partial_t w - \delta_\Gamma \Delta_\Gamma w &= \frac{1}{\delta_{k'}} z - \frac{1}{\delta_k} uw && \text{on } \Gamma \\
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$$(u(0), w(0), z(0)) = (u_0, w_0, z_0).$$

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Asymptotic limits

- Biologically relevant scales:

$$0 < 1/\delta_{k'} \ll 1$$

- We set $1/\delta_{k'} = 0$ and hence neglected z.
- Existence and uniqueness.
- Convergence to limiting free boundary problems for the cases
  1. $\delta_k \to 0$
  2. $\delta_k = \delta_\Gamma \to 0$
  3. $\delta_k = \delta_\Gamma = \delta_\Omega \to 0$
- Formulation of the limits as Stefan-type surface FBPs.
- Uniqueness for the limiting FBPs (2 and 3).
- Numerics for original problem and limiting FBPs (2 and 3).

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Coupled bulk-surface model for receptor-ligand dynamics (Elliott, Ranner, and Venkataraman, 2017)

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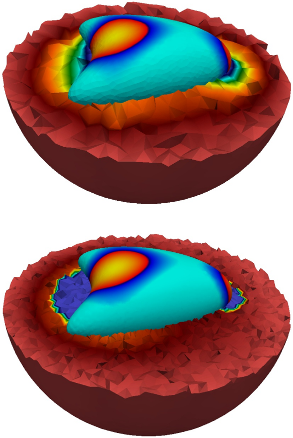

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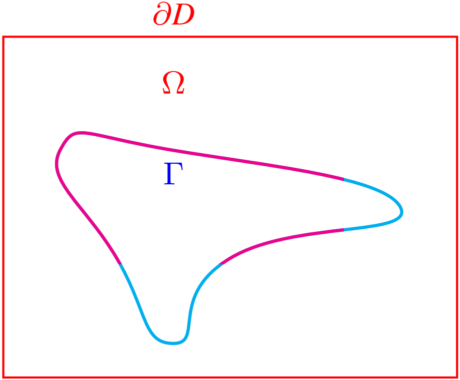
Fast binding limits

The asymptotic limits $(\delta_k \to 0)$ lead to interesting FBPs due to the complementarity nature of the resulting limit:

$$u \geq 0, \qquad w \geq 0, \qquad uw = 0 \qquad \text{on } \Gamma.$$



"Perfectly reflecting or perfectly monitoring"

$w = 0, \quad u > 0$ and $\nabla u \cdot \boldsymbol{\nu} = 0$

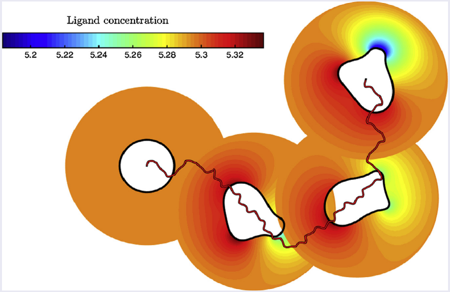
"Perfectly absorbing"

$u = 0$ and $w > 0$

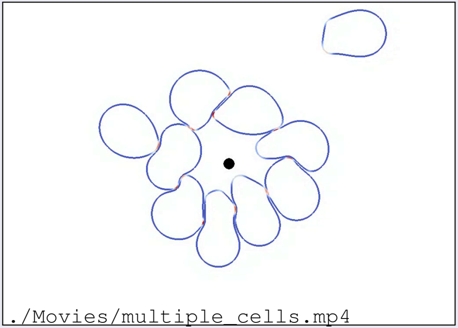
(Endres and Wingreen, 2008)

Systems of coupled bulk-surface PDEs on evolving domains arise in a number of models for phenomena in cell biology such as chemotaxis, adhesion, signalling, . . .

Chemotaxis



Simulation of single-cell chemotaxis (MacDonald et al., 2016)



Simulation of multi-cell chemotaxis (Elliott, Stinner, and Venkataraman, 2012)

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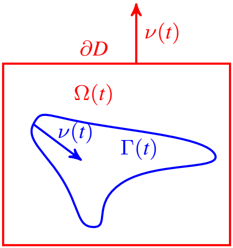
Evolving domain coupled bulk-surface model for receptor-ligand dynamics (Alphonse, Elliott, and Terra, 2018)



- $\mathbf{V}_\Omega$: material velocity of Ω
- $\mathbf{V}_\Gamma$: material velocity of Γ
- $\mathbf{V}_p$: parametrisation velocity field.

Model

$$\mathbf{J}_\Omega := \mathbf{V}_\Omega - \mathbf{V}_p, \qquad \mathbf{J}_\Gamma := \mathbf{V}_\Gamma - \mathbf{V}_p \text{ and } j := (\mathbf{V}_\Omega - \mathbf{V}_\Gamma) \cdot \nu.$$

$$\partial^\bullet w := w_t + \nabla w \cdot \mathbf{V}_p$$

$$\delta_\Omega(\partial^\bullet u + u\nabla \cdot \mathbf{V}_p + \nabla \cdot (\mathbf{J}_\Omega u)) - \Delta u = 0 \qquad \text{in } \Omega(t)$$

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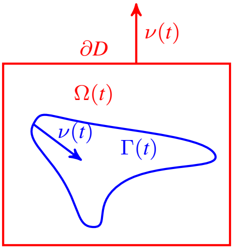
$\mathbf{V}_p$: not intrinsic to the model but used to define function spaces and relevant to numerics.
$j \neq 0$ corresponds to the so called "windshield effect".

Evolving domain coupled bulk-surface model for receptor-ligand dynamics (Alphonse, Elliott, and Terra, 2018)



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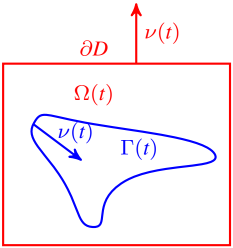

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Evolving domain coupled bulk-surface model for receptor-ligand dynamics (Alphonse, Elliott, and Terra, 2018)

Theorem (Existence & Uniqueness)

For bounded non-negative data and smooth velocity fields, there exists a unique non-negative weak solution to the evolving domain receptor-ligand system.

- Analysis based on the abstract evolving function space framework of (Alphonse, Elliott, and Stinner, 2015a,b; Vierling, 2014)
- Asymptotic limits and numerics not considered.

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Asymptotic limits of an evolving domain coupled bulk-surface model for receptor-ligand dynamics

Model

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\partial^\bullet z + z\nabla_\Gamma\cdot\mathbf{V}_p - \delta_{\Gamma'}\Delta_\Gamma z + \nabla_\Gamma\cdot(\mathbf{J}_\Gamma z) = \frac{1}{\delta_k} g(u,w) - \frac{1}{\delta_{k'}} z & \qquad \text{on } \Gamma(t) \\
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\end{aligned}$$

$$g(u,w) = uw \qquad \text{or} \qquad g(u,w) = \frac{u^n w}{1+u^n} \text{ for } n > 1.$$

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\end{aligned}
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Assumption: Existence & Uniqueness

We take non-negative bounded data and we *assume* the existence of a unique non-negative solution.
(Satisfied in the case $g(u,w) = uw$ from the results of (Alphonse, Elliott, and Terra, 2018))

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Fast binding limit: $\delta_k \to 0$

As $\delta_k \to 0$, the solution (u_k, w_k, z_k) converges to a weak solution (u, w, z) of the problem

$$
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g(u,w) &= 0 && \text{on } \Gamma(t)\\
(u(0), w(0), z(0)) &= (u_0, w_0, z_0).
\end{aligned}
$$

A weak solution is defined to satisfy, for every $\eta \in \mathcal{V}$ with $\eta(T) = 0$,

$$
\begin{aligned}
&-\int_{\Omega_0} u_0\eta(0) + \frac{1}{\delta_\Omega}\int_{\Gamma_0} w_0\eta(0) - \int_0^T\int_{\Omega(t)} u\partial^\bullet\eta + \frac{1}{\delta_\Omega}\int_0^T\int_\Gamma w\partial^\bullet\eta + \frac{1}{\delta_\Omega}\int_0^T\int_{\Omega(t)}\nabla u\cdot\nabla\eta\\
&-\frac{\delta_\Gamma}{\delta_\Omega}\int_0^T\int_\Gamma \nabla_\Gamma w\cdot\nabla_\Gamma\eta + \int_0^T\int_{\Omega(t)}\nabla\cdot(\mathbf{J}_\Omega u)\eta - \frac{1}{\delta_\Omega}\int_0^T\int_{\Gamma(t)}\nabla_\Gamma\cdot(\mathbf{J}_\Gamma w)\eta - \int_0^T\int_{\Gamma(t)} ju\eta = 0,\\
&-\int_{\Omega_0} u_0\eta(0) - \frac{1}{\delta_\Omega}\int_{\Gamma_0} z_0\eta(0) - \int_0^T\int_{\Omega(t)} u\partial^\bullet\eta - \frac{1}{\delta_\Omega}\int_0^T\int_\Gamma z\partial^\bullet\eta + \frac{1}{\delta_\Omega}\int_0^T\int_{\Omega(t)}\nabla u\cdot\nabla\eta\\
&+\frac{\delta_{\Gamma'}}{\delta_\Omega}\int_0^T\int_\Gamma \nabla_\Gamma z\cdot\nabla_\Gamma\eta + \int_0^T\int_{\Omega(t)}\nabla\cdot(\mathbf{J}_\Omega u)\eta + \frac{1}{\delta_\Omega}\int_0^T\int_{\Gamma(t)}\nabla_\Gamma\cdot(\mathbf{J}_\Gamma z)\eta - \int_0^T\int_{\Gamma(t)} ju\eta = 0,
\end{aligned}
$$

$g(u, w) = 0$ on $\Gamma(t)$.

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Fast binding limit: $\delta_k \to 0$

As $\delta_k \to 0$, the solution (u_k, w_k, z_k) converges to a weak solution (u, w, z) of the problem

$$\delta_\Omega(\partial^\bullet u + u\nabla\cdot\mathbf{V}_p + \nabla\cdot(\mathbf{J}_\Omega u)) - \Delta u = 0 \qquad \text{in } \Omega(t)$$
$$\nabla u\cdot\nu = 0 \qquad \text{on } \partial D$$
$$\partial^\bullet w - \delta_\Gamma\Delta_\Gamma w + w\nabla_\Gamma\cdot\mathbf{V}_p + \nabla_\Gamma\cdot(\mathbf{J}_\Gamma w) = \nabla u\cdot\nu - \delta_\Omega ju \qquad \text{on } \Gamma(t)$$
$$\partial^\bullet z - \delta_{\Gamma'}\Delta_\Gamma z + z\nabla_\Gamma\cdot\mathbf{V}_p + \nabla_\Gamma\cdot(\mathbf{J}_\Gamma z) = -\nabla u\cdot\nu + \delta_\Omega ju \qquad \text{on } \Gamma(t)$$
$$g(u,w) = 0 \qquad \text{on } \Gamma(t)$$
$$(u(0), w(0), z(0)) = (u_0, w_0, z_0).$$

A weak solution is defined to satisfy, for every $\eta\in\mathcal{V}$ with $\eta(T) = 0$,

$$-\int_{\Omega_0} u_0\eta(0) + \frac{1}{\delta_\Omega}\int_{\Gamma_0} w_0\eta(0) - \int_0^T\int_{\Omega(t)} u\partial^\bullet\eta + \frac{1}{\delta_\Omega}\int_0^T\int_\Gamma w\partial^\bullet\eta + \frac{1}{\delta_\Omega}\int_0^T\int_{\Omega(t)}\nabla u\cdot\nabla\eta$$
$$-\frac{\delta_\Gamma}{\delta_\Omega}\int_0^T\int_\Gamma\nabla_\Gamma w\cdot\nabla_\Gamma\eta + \int_0^T\int_{\Omega(t)}\nabla\cdot(\mathbf{J}_\Omega u)\eta - \frac{1}{\delta_\Omega}\int_0^T\int_{\Gamma(t)}\nabla_\Gamma\cdot(\mathbf{J}_\Gamma w)\eta - \int_0^T\int_{\Gamma(t)} ju\eta = 0,$$
$$-\int_{\Omega_0} u_0\eta(0) - \frac{1}{\delta_\Omega}\int_{\Gamma_0} z_0\eta(0) - \int_0^T\int_{\Omega(t)} u\partial^\bullet\eta - \frac{1}{\delta_\Omega}\int_0^T\int_\Gamma z\partial^\bullet\eta + \frac{1}{\delta_\Omega}\int_0^T\int_{\Omega(t)}\nabla u\cdot\nabla\eta$$
$$+\frac{\delta_{\Gamma'}}{\delta_\Omega}\int_0^T\int_\Gamma\nabla_\Gamma z\cdot\nabla_\Gamma\eta + \int_0^T\int_{\Omega(t)}\nabla\cdot(\mathbf{J}_\Omega u)\eta + \frac{1}{\delta_\Omega}\int_0^T\int_{\Gamma(t)}\nabla_\Gamma\cdot(\mathbf{J}_\Gamma z)\eta - \int_0^T\int_{\Gamma(t)} ju\eta = 0,$$

$g(u,w) = 0$ on $\Gamma(t)$.

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Fast binding, slow surface diffusion limit: $\delta_k = \delta_\Gamma = \delta_{\Gamma'} \to 0$

As $\delta_k = \delta_\Gamma = \delta_{\Gamma'} \to 0$, the solution (u_k, w_k, z_k) converges to a weak solution (u, w, z) of the problem

$$
\begin{aligned}
\delta_\Omega(\partial^\bullet u + u\nabla_\Gamma \cdot \mathbf{V}_p + \nabla \cdot (\mathbf{J}_\Omega u)) - \Delta u &= 0 && \text{in } \Omega(t)\\
\nabla u \cdot \nu &= 0 && \text{on } \partial D\\
\partial^\bullet w + w\nabla_\Gamma \cdot \mathbf{V}_p + \nabla_\Gamma \cdot (\mathbf{J}_\Gamma w) &= \nabla u \cdot \nu - \delta_\Omega j u && \text{on } \Gamma(t)\\
\partial^\bullet z + z\nabla_\Gamma \cdot \mathbf{V}_p + \nabla_\Gamma \cdot (\mathbf{J}_\Gamma z) &= -\nabla u \cdot \nu + \delta_\Omega j u && \text{on } \Gamma(t)\\
g(u, w) &= 0 && \text{on } \Gamma(t)\\
(u(0), w(0), z(0)) &= (u_0, w_0, z_0).
\end{aligned}
$$

A weak solution is defined to satisfy, for every $\eta \in \mathcal{V}$ with $\eta(T) = 0$,

$$
\begin{aligned}
&-\int_{\Omega_0} u_0\psi(0) + \frac{1}{\delta_\Omega}\int_{\Gamma_0} w_0\psi(0) - \int_0^T\int_{\Omega(t)} u\partial^\bullet\eta + \frac{1}{\delta_\Omega}\int_0^T\int_{\Gamma(t)} w\partial^\bullet\eta + \frac{1}{\delta_\Omega}\int_0^T\int_{\Omega(t)} \nabla u \cdot \nabla\eta\\
&\quad + \int_0^T\int_{\Omega(t)} \nabla \cdot (\mathbf{J}_\Omega u)\eta + \frac{1}{\delta_\Omega}\int_0^T\int_{\Gamma(t)} w\mathbf{J}_\Gamma \cdot \nabla_\Gamma\eta - \int_0^T\int_{\Gamma(t)} ju\eta = 0,\\
&-\int_{\Omega_0} u_0\psi(0) - \frac{1}{\delta_\Omega}\int_{\Gamma_0} z_0\psi(0) - \int_0^T\int_{\Omega(t)} u\partial^\bullet\eta - \frac{1}{\delta_\Omega}\int_0^T\int_{\Gamma(t)} z\partial^\bullet\eta + \frac{1}{\delta_\Omega}\int_0^T\int_{\Omega(t)} \nabla u \cdot \nabla\eta\\
&\quad + \int_0^T\int_{\Omega(t)} \nabla \cdot (\mathbf{J}_\Omega u)\eta - \frac{1}{\delta_\Omega}\int_0^T\int_{\Gamma(t)} z\mathbf{J}_\Gamma \cdot \nabla_\Gamma\eta - \int_0^T\int_{\Gamma(t)} ju\eta = 0,\\
&g(u, w) = 0 \text{ on } \Gamma(t).
\end{aligned}
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Fast binding, slow surface diffusion limit: $\delta_k = \delta_\Gamma = \delta_{\Gamma'} \to 0$

As $\delta_k = \delta_\Gamma = \delta_{\Gamma'} \to 0$, the solution (u_k, w_k, z_k) converges to a weak solution (u, w, z) of the problem

$$
\begin{aligned}
\delta_\Omega(\partial^\bullet u + u\nabla_\Gamma \cdot \mathbf{V}_p + \nabla \cdot (\mathbf{J}_\Omega u)) - \Delta u &= 0 && \text{in } \Omega(t)\\
\nabla u \cdot \nu &= 0 && \text{on } \partial D\\
\partial^\bullet w + w\nabla_\Gamma \cdot \mathbf{V}_p + \nabla_\Gamma \cdot (\mathbf{J}_\Gamma w) &= \nabla u \cdot \nu - \delta_\Omega j u && \text{on } \Gamma(t)\\
\partial^\bullet z + z\nabla_\Gamma \cdot \mathbf{V}_p + \nabla_\Gamma \cdot (\mathbf{J}_\Gamma z) &= -\nabla u \cdot \nu + \delta_\Omega j u && \text{on } \Gamma(t)\\
g(u, w) &= 0 && \text{on } \Gamma(t)\\
(u(0), w(0), z(0)) &= (u_0, w_0, z_0).
\end{aligned}
$$

A weak solution is defined to satisfy, for every $\eta \in \mathcal{V}$ with $\eta(T) = 0$,

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\begin{aligned}
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&\quad + \int_0^T\int_{\Omega(t)} \nabla\cdot(\mathbf{J}_\Omega u)\eta + \frac{1}{\delta_\Omega}\int_0^T\int_{\Gamma(t)} w\mathbf{J}_\Gamma\cdot\nabla_\Gamma\eta - \int_0^T\int_{\Gamma(t)} ju\eta = 0,\\
&-\int_{\Omega_0} u_0\psi(0) - \frac{1}{\delta_\Omega}\int_{\Gamma_0} z_0\psi(0) - \int_0^T\int_{\Omega(t)} u\partial^\bullet\eta - \frac{1}{\delta_\Omega}\int_0^T\int_{\Gamma(t)} z\partial^\bullet\eta + \frac{1}{\delta_\Omega}\int_0^T\int_{\Omega(t)} \nabla u\cdot\nabla\eta\\
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\end{aligned}
$$

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Assume that $\mathbf{J}_\Gamma \equiv 0$. As $\delta_\Omega = \delta_k = \frac{1}{\delta_{k'}} = \delta_\Gamma = \delta_{\Gamma'} \to 0$, the solution (u_k, w_k, z_k) converges to a weak solution (u, w, z) of the problem

$$
\begin{aligned}
\Delta u &= 0 && \text{in } \Omega(t)\\
u &= u_D && \text{on } \partial D\\
\partial^\bullet w + w\nabla_\Gamma \cdot \mathbf{V}_\Gamma &= \nabla u \cdot \nu && \text{on } \Gamma(t)\\
\partial^\bullet z + z\nabla_\Gamma \cdot \mathbf{V}_\Gamma &= -\nabla u \cdot \nu && \text{on } \Gamma(t)\\
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(w(0), z(0)) &= (w_0, z_0).
\end{aligned}
$$

A weak solution is defined to satisfy for every $\eta \in \mathcal{V}_e$ with $\eta(T) = 0$

$$
\int_0^T \int_{\Omega(t)} \nabla u \cdot \nabla \eta + \int_{\Gamma_0} w_0 \eta(0) + \int_0^T \int_{\Gamma(t)} w \partial^\bullet \eta = 0
$$

$$
\int_0^T \int_{\Omega(t)} \nabla u \cdot \nabla \eta - \int_{\Gamma_0} z_0 \eta(0) - \int_0^T \int_{\Gamma(t)} z \partial^\bullet \eta = 0
$$

$u = u_D$ on ∂D

$g(u, w) = 0$ on $\Gamma(t)$.

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$\delta_k = \delta_\Gamma = \delta_{\Gamma'} = \delta_\Omega = \delta_{k'}^{-1} \to 0$ limit

Assume that $\mathbf{J}_\Gamma \equiv 0$. As $\delta_\Omega = \delta_k = \frac{1}{\delta_{k'}} = \delta_\Gamma = \delta_{\Gamma'} \to 0$, the solution (u_k, w_k, z_k) converges to a weak solution (u, w, z) of the problem

$$\begin{aligned} \Delta u &= 0 && \text{in } \Omega(t) \\ u &= u_D && \text{on } \partial D \\ \partial^\bullet w + w \nabla_\Gamma \cdot \mathbf{V}_\Gamma &= \nabla u \cdot \nu && \text{on } \Gamma(t) \\ \partial^\bullet z + z \nabla_\Gamma \cdot \mathbf{V}_\Gamma &= -\nabla u \cdot \nu && \text{on } \Gamma(t) \\ g(u, w) &= 0 && \text{on } \Gamma(t) \\ (w(0), z(0)) &= (w_0, z_0). \end{aligned}$$

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$$\int_0^T \int_{\Omega(t)} \nabla u \cdot \nabla \eta - \int_{\Gamma_0} z_0 \eta(0) - \int_0^T \int_{\Gamma(t)} z \partial^\bullet \eta = 0$$

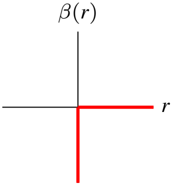
$$u = u_D \text{ on } \partial D$$

$$g(u, w) = 0 \text{ on } \Gamma(t).$$

Formulation as Stefan-type free boundary problems on evolving surfaces

If we define $v := -w$ and define the maximal monotone graph β by

$$\beta(r) := \begin{cases} \emptyset & : r < 0 \\ (-\infty, 0] & : r = 0 \\ 0 & : r > 0 \end{cases}$$



we can recast the condition $u, w \geq 0, g(u, w) = 0$ on $\Gamma(t)$ as $v \in \beta(u)$ on $\Gamma(t)$.

Defining $P\colon L^2_{H^{1/2}(\Gamma)} \to L^2_{H^1(\Omega)}$ the solution map $u \mapsto U$ of

$$\begin{aligned} \delta_\Omega(\partial^\bullet U + U\nabla \cdot \mathbf{V}_p + \nabla \cdot (\mathbf{J}_\Omega U)) - \Delta U &= 0 && \text{in } \Omega(t) \\ \nabla U \cdot \nu &= 0 && \text{on } \partial D \\ U &= u && \text{on } \Gamma(t) \\ U(0) &= u_0 && \text{on } \Omega_0 \end{aligned}$$

and define the associated Dirichlet-to-Neumann map $A\colon L^2_{H^{1/2}(\Gamma)} \to L^2_{H^{-1/2}(\Gamma)}$ by

$$A(u) := \nabla P(u) \cdot \nu.$$

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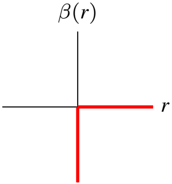

we can recast the condition $u, w \geq 0, g(u, w) = 0$ on $\Gamma(t)$ as $v \in \beta(u)$ on $\Gamma(t)$.

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Formulation as Stefan-type free boundary problems on evolving surfaces

$\delta_k \to 0$ limit. We can rewrite the problem as: find (u, v) with $u \geq 0$, $v \leq 0$, such that

$$\partial^\bullet v - \delta_\Gamma \Delta_\Gamma v + v\nabla_\Gamma \cdot \mathbf{V}_p + \nabla_\Gamma \cdot (\mathbf{J}_\Gamma v) + A(u) = \delta_\Omega j u \qquad \text{on } \Gamma(t)$$
$$v \in \beta(u)$$
$$v(0) = v_0$$

$\delta_k = \delta_\Gamma = \delta_{\Gamma'} \to 0$ limit. We can rewrite the problem as: find (u, v) with $u \geq 0$, $v \leq 0$, such that

$$\partial^\bullet v + v\nabla_\Gamma \cdot \mathbf{V}_p + \nabla_\Gamma \cdot (\mathbf{J}_\Gamma v) + A(u) = \delta_\Omega j u \qquad \text{on } \Gamma(t)$$
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$\delta_\Omega = \delta_k = \delta_{k'}^{-1} = \delta_\Gamma = \delta_{\Gamma'} \to 0$ limit . We can rewrite the problem as: find (u, v) with $u \geq 0$, $v \leq 0$ such that

$$\partial^\bullet v + v\nabla_\Gamma \cdot \mathbf{V}_\Gamma + A_D(u) + \nabla h \cdot \nu = 0 \qquad \text{on } \Gamma(t)$$
$$v \in \beta(u)$$
$$v(0) = v_0$$

Approach for uniqueness used in fixed domain setting (Elliott, Ranner, and Venkataraman, 2017) *not* directly applicable.

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Formulation as Stefan-type free boundary problems on evolving surfaces

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$$v \in \beta(u)$$
$$v(0) = v_0$$

Approach for uniqueness used in fixed domain setting (Elliott, Ranner, and Venkataraman, 2017) *not* directly applicable.

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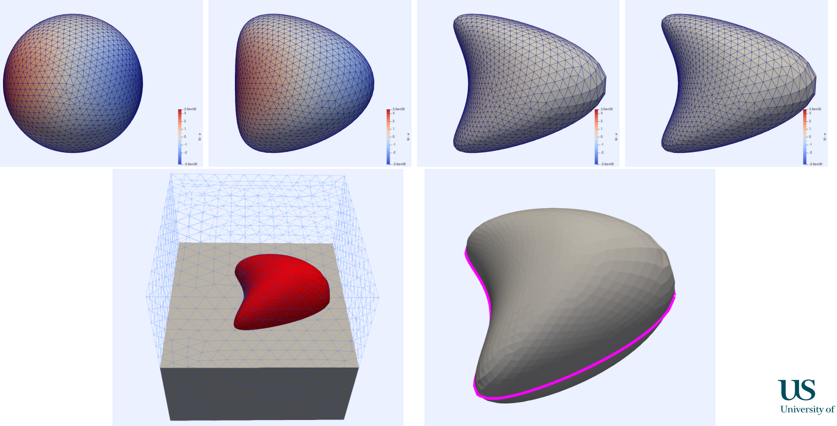
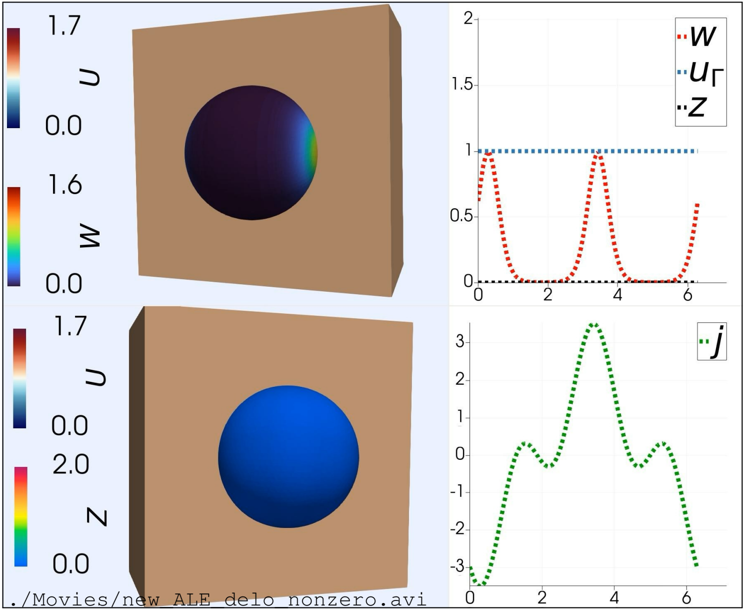
Numerics

- $D = [-2, 2]^3$
- $\Gamma(t)$ defined as the zero level set of $\phi(\boldsymbol{x}, t) = \left(x_1 + \tanh(5t)\left(0.7 - x_2^2\right)\right)^2 + x_2^2 + x_3^2 - 1.$
- We set $\mathbf{V}_\Gamma = V\boldsymbol{\nu}$.
- We take $\mathbf{V}_\Omega = \mathbf{0}$ (windshield effect) or $\mathbf{V}_\Omega = E(\mathbf{V}_\Gamma)$ (no windshield effect)
- We set $u_0 = u_D = 1, z_0 = 0$
- Numerics based on an evolving coupled bulk-surface FEM.



$$\delta_k = \delta_\Gamma = \delta_{\Gamma'} = 0.001, \qquad \mathbf{V}_\Omega = 0, \qquad g(u,w) = \frac{u^2}{1+u^2} w$$



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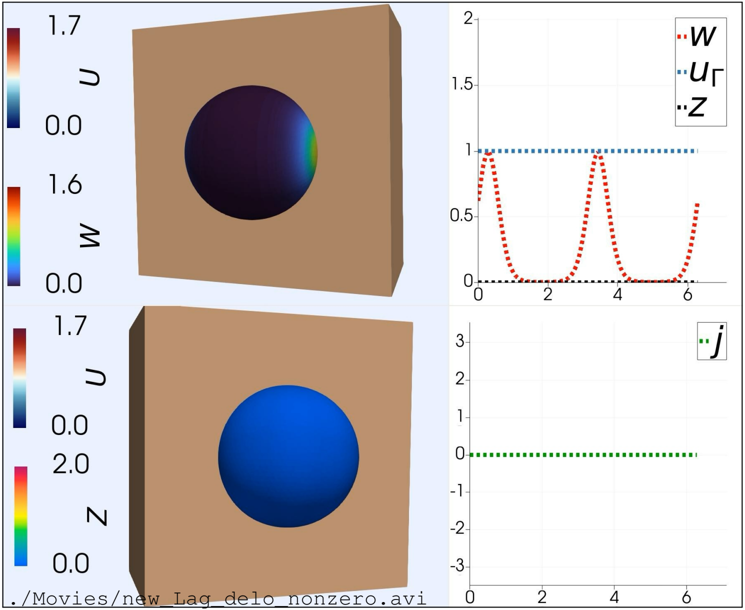
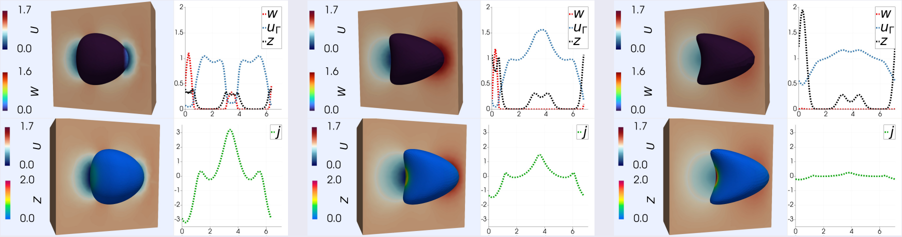
$\delta_k = \delta_\Gamma = \delta_{\Gamma'} = 0.001, \qquad \mathbf{V}_\Omega = E(\mathbf{V}_\Gamma), \qquad g(u,w) = \frac{u^2}{1+u^2} w$

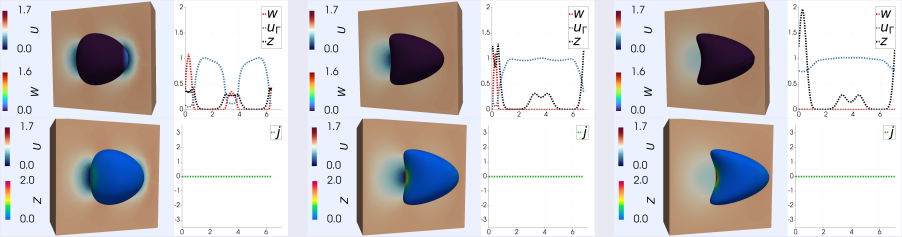


$\mathbf{V}_\Omega = 0$



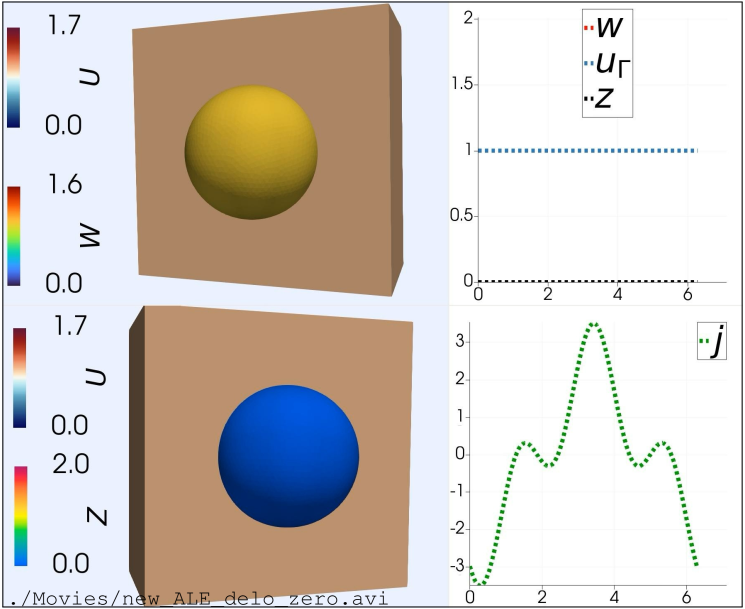
(a) $t = 0.06$ (b) $t = 0.2$ (c) $t = 0.4$

$\mathbf{V}_\Omega = E(\mathbf{V}_\Gamma)$

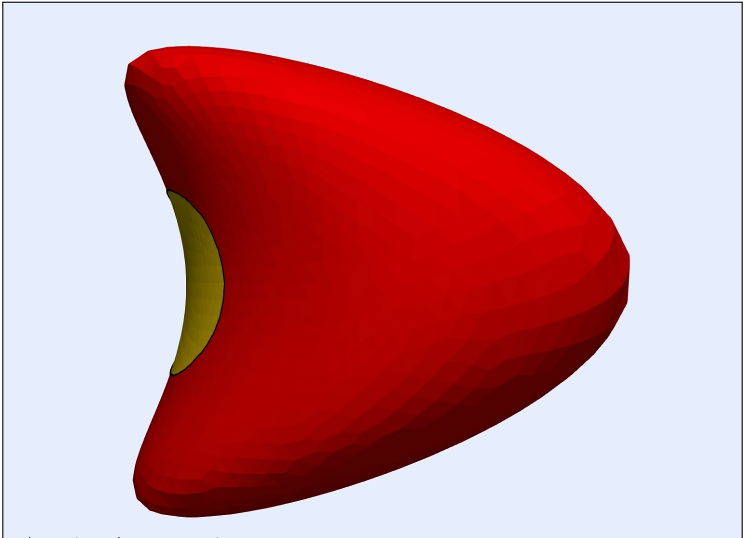


(d) $t = 0.06$ (e) $t = 0.2$ (f) $t = 0.4$

$$\delta_k = \delta_\Gamma = \delta_{\Gamma'} = \delta_\Omega = \delta_{k'}^{-1} = 0.01, \qquad \mathbf{V}_\Omega = 0, \qquad g(u,w) = uw$$



Evolving Surface FBP



./Movies/FBP_movie.mov

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Summary & Outlook

Summary

- Asymptotic limits of models for receptor-ligand interactions on *evolving domains*.
- Limiting problems correspond to Stefan-type free boundary problems on an evolving surface.
- Numerics illustrating the free boundary and effects due to domain evolution.

Outlook

- Uniqueness for the limiting problems (vanishing surface diffusion cases).
- Numerics for the limiting problems (vanishing surface diffusion cases).
- Coupling to biochemistry in cell bulk.
- Coupling to (unknown) surface evolution.

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References:

CM Elliott, T Ranner and C Venkataraman. "Coupled bulk-surface free boundary problems arising from a mathematical model of receptor-ligand dynamics". SIAM J. Math. Anal. 2017

A Alphonse, CM Elliott, and J Terra. "A coupled ligand-receptor bulk-surface system on a moving domain : well posedness, regularity and convergence to equilibrium". SIAM J. Math. Anal. 2018

A Alphonse, D Caetano, CM Elliott, and C Venkataraman. "Free boundary limits of coupled bulk surface models for receptor-ligand interactions on evolving domains". *in prep*

Thank You