Mathematics Institute
University of Warwick

Convergent finite element schemes with mesh smoothing for geometrically evolving curves and networks

Björn Stinner
(joint work with Paola Pozzi)

Mathematical Aspects for Interfaces and Free Boundaries
81st Fujihara Seminar, June 2024

Motivation

Context:

Parametric numerical (finite element) approaches for geometric evolution equations.
[Brakke 1978], [Dziuk 1991, 1994], [Deckelnick, Dziuk 1994+],
[Bronsard, Wetton 1993], [Walkington 1996],
[Mayer, Simonett 2002], [Bänsch, Morin, Nochetto 2005],
[Clarenz, Diewald, Dziuk, Rumpf, Rusu 2004],
[Barrett, Garcke, Nürnberg 2008+], [Mikula, Ševčovič 2001+],
[Elliott, Fritz 2016], [Kovács, Li, Lubich 2019+].

Problem:

Normal motion $\rightsquigarrow$ mesh degeneration.

Adding tangential movement?

- $+$ Beneficial for long-term computations (possibly also the analysis).
- $-$ Might lose structure (gradient flow, variational).

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[Elliott, Fritz 2016], [Kovács, Li, Lubich 2019+].

Idea:

Use (variants of) the Dirichlet energy
and augment the system energy / replace it.

Objectives:

- Keep variational structure $\rightsquigarrow$ error analysis.
- Minimise the impact on the system's physics.

Two examples:

1. Triods (simple network) subject to motion by curve shortening flow.
2. Relaxation of the elastic energy of closed curves.

Outline

Geometrically evolving triods minimising the network length

Relaxing the elastic energy of a closed curve

Evolving triod

Geometric problem:

Triod, three curves,

moving by curvature,

120° angles
at triple junction,

end points fixed.

evolving triod

Objective: parametrisation,
formulate in a variational form amenable to FEs, and
prove convergence.

First attempt

Single curve: $\tilde{u} : [0,1] \times (0,T) \to \mathbb{R}^2$,

$$\tilde{u}_t = \partial_{ss}\tilde{u} = \frac{1}{|\tilde{u}_x|}\frac{d}{dx}\Big(\frac{\tilde{u}_x}{|\tilde{u}_x|}\Big).$$

Variational form, sum for curves forming a triod:

$$\sum_{i=1}^{3}\int_0^1 u_t^{(i)} \cdot \phi^{(i)}|u_x^{(i)}| + \frac{u_x^{(i)}}{|u_x^{(i)}|} \cdot \phi_x^{(i)} dx = 0.$$

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Benefits:

- Gradient flow of $\sum_i \int_0^1 |u_x^{(i)}| dx$ (in some sense),
- can be parametrised by standard linear Lagrange FEs,
- error analysis for single curves ([Dziuk 1994], [Pozzi 2007]),
- angle condition correctly accounted for,

$$0 = \sum_{i=1}^{3} \tau^{(i)} = \sum_{i=1}^{3} \frac{u_x^{(i)}}{|u_x^{(i)}|}.$$

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Problem:

Movement of points on the curves purely in normal direction, hence the triple junction is immobile.

[Mantegazza, Novaga, Pluda, Schulze 2016]

Second attempt

Single curve [Deckelnick, Dziuk 1994]
(reparametrisation with harmonic map flow, [Elliott, Fritz 2016]):

$$\tilde{u}_t |\tilde{u}_x|^2 = \tilde{u}_{xx} \quad \text{(variation of the Dirichlet energy } \int_0^1 \tfrac{1}{2} |\tilde{u}_x|^2 dx \text{).}$$

Variational form, for curve, sum for triod:

$$\sum_{i=1}^{3} \int_0^1 u_t^{(i)} \cdot \phi^{(i)} |u_x^{(i)}|^2 dx + u_x^{(i)} \cdot \phi_x^{(i)} dx = 0.$$

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- Allows for triple junction movement,
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- used for computations
  [Bronsard, Wetton 1993], [Deckelnick, Elliott 1998], [Pan, Wetton 2012].

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Problem:
Triple junction condition not correctly implemented,

$$\text{we have } 0 = \sum_{i=1}^{3} u_x^{(i)} \qquad \text{but we want } 0 = \sum_{i=1}^{3} \frac{u_x^{(i)}}{|u_x^{(i)}|}.$$

Combination

Idea: first attempt in normal direction,
second attempt in tangential direction but scaled ($\varepsilon > 0$):

$$\sum_{i=1}^{3}\Bigg(\int_0^1 (u_t^{(i)}\cdot\nu^{(i)})(\varphi^{(i)}\cdot\nu^{(i)})|u_x^{(i)}|dx + \epsilon\int_0^1 (u_t^{(i)}\cdot\tau^{(i)})(\varphi^{(i)}\cdot\tau^{(i)})|u_x^{(i)}|^2 dx\Bigg) = -\sum_{i=1}^{3}\Bigg(\epsilon\int_0^1 u_x^{(i)}\cdot\varphi_x^{(i)}dx + \int_0^1 \tau^{(i)}\cdot\varphi_x^{(i)}dx\Bigg).$$

Can be approximated using linear Lagrange finite elements.

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Can be approximated using linear Lagrange finite elements.

Strong form (desired model modulo ϵ perturbation):

$$(u_t^{(i)}\cdot\nu^{(i)})\nu^{(i)} = (1+\epsilon|u_x^{(i)}|)\kappa^{(i)},$$

$$(u_t^{(i)}\cdot\tau^{(i)})\tau^{(i)} = \frac{1}{|u_x^{(i)}|^2}(\tau^{(i)}\cdot u_{xx})\tau^{(i)},$$

$$0 = \sum_{i=1}^{3}\tau^{(i)} + \epsilon u_x^{(i)}.$$

Theorem

Convergence and error estimate [Pozzi, S, SMAI JCM 2021]

(h-convergence, for ε fixed)

Assume that there is a unique (sufficiently regular) solution $\Gamma = (u^{(1)}, u^{(2)}, u^{(3)})$ with

$$0 < c_0 \leq |u_x^{(i)}| \leq 1/c_0.$$

For all h small enough the semi-discrete problem has a unique solution $\Gamma_h = (u_h^{(1)}, u_h^{(2)}, u_h^{(3)})$ satisfying

$$\int_0^T \|u_t^{(i)} - u_{ht}^{(i)}\|_{L^2(\Omega)}^2 dt + \sup_{t \in [0,T]} \|u_x^{(i)}(t) - u_{hx}^{(i)}(t)\|_{L^2(\Omega)}^2 \leq Ch^2, \quad i = 1, 2, 3.$$

The constant $C > 0$ depends on c_0, T, norms of the $u^{(i)}$, and scales with ϵ^{-1}.

Proof

Following [Deckelnick, Dziuk 1994].

Fixed point argument on

$$\mathcal{B}_h := \left\{ \Gamma_h = (u_h^{(1)}, u_h^{(2)}, u_h^{(3)}) \,\middle|\, \text{..., admissible triods, ...} \right.$$
$$\left. \sup_{t\in[0,T]} e^{-Mt} \|(u_x^{(i)} - u_{hx}^{(i)})(t)\|^2_{L^2(\Omega)} \leq K^2 h^2 \, \forall i \right\}.$$

Fixed point map: Given $\Gamma_h = (u_h^{(1)}, u_h^{(2)}, u_h^{(3)}) \in \mathcal{B}_h$,
find $(Y_h^{(1)}(t), Y_h^{(2)}(t), Y_h^{(3)}(t))$ such that

$$\sum_{i=1}^{3} \left(\int_\Omega (Y_{ht}^{(i)} \cdot \frac{(u_{hx}^{(i)})^\perp}{|u_{hx}^{(i)}|})(\varphi_h^{(i)} \cdot \frac{(u_{hx}^{(i)})^\perp}{|u_{hx}^{(i)}|})|u_{hx}^{(i)}| dx \right.$$
$$\left. + \epsilon \int_\Omega (Y_{ht}^{(i)} \cdot \frac{u_{hx}^{(i)}}{|u_{hx}^{(i)}|})(\varphi_h^{(i)} \cdot \frac{u_{hx}^{(i)}}{|u_{hx}^{(i)}|})|u_{hx}^{(i)}|^2 dx \right)$$
$$= -\sum_{i=1}^{3} \left(\epsilon \int_\Omega Y_{hx}^{(i)} \cdot \varphi_{hx}^{(i)} dx + \int_\Omega \frac{Y_{hx}^{(i)}}{|Y_{hx}^{(i)}|} \cdot \varphi_{hx}^{(i)} dx \right).$$

Note that $0 < c_0/2 \leq |u_{hx}^{(i)}| \leq 2/c_0$ for h small enough.

Proof

Following [Deckelnick, Dziuk 1994].

Fixed point argument on

$$\mathcal{B}_h := \{\, \Gamma_h = (u_h^{(1)}, u_h^{(2)}, u_h^{(3)}) \mid \text{ ..., admissible triods, ...} \\ \sup_{t\in[0,T]} e^{-Mt} \|(u_x^{(i)} - u_{hx}^{(i)})(t)\|_{L^2(\Omega)}^2 \le K^2 h^2 \,\forall i \,\}.$$

Proposition: For h small enough there is a unique solution $(Y_h^{(1)}, Y_h^{(2)}, Y_h^{(3)})$ that satisfies the estimates

$$\sup_{t\in[0,T]} e^{-Mt} \|u_x^{(i)}(t) - Y_{hx}^{(i)}(t)\|_{L^2(\Omega)}^2 \le \Big(1 + \frac{K^2}{M}\Big) C h^2,$$

$$\int_0^T \|u_t^{(i)}(t') - Y_{ht}^{(i)}(t')\|_{L^2(\Omega)}^2 dt' \le \tilde{C} h^2,$$

$C = C(c_0, T, \epsilon, C_p, \text{ norms of the } u^{(i)}) > 0$,
$\tilde{C} > 0$ depending on the same parameters and M and K.

Numerical assessment

Simple first order IMEX time discretisation, $\delta = T/N > 0$.

$$\sum_{i=1}^{3}\Bigg(\int_\Omega\Big(\frac{U^{(i),n}-U^{(i),n-1}}{\delta}\cdot\frac{(U_x^{(i),n-1})^\perp}{|U_x^{(i),n-1}|}\Big)\Big(\varphi_h^{(i)}\cdot\frac{(U_x^{(i),n-1})^\perp}{|U_x^{(i),n-1}|}\Big)|U_x^{(i),n-1}|dx$$

$$+\,\epsilon\int_\Omega\Big(\frac{U^{(i),n}-U^{(i),n-1}}{\delta}\cdot\frac{U_x^{(i),n-1}}{|U_x^{(i),n-1}|}\Big)\Big(\varphi_h^{(i)}\cdot\frac{U_x^{(i),n-1}}{|U_x^{(i),n-1}|}\Big)|U_x^{(i),n-1}|^2dx\Bigg)$$

$$+\sum_{i=1}^{3}\left(\epsilon\int_\Omega U_x^{(i),n}\cdot\varphi_{hx}^{(i)}dx+\int_\Omega\frac{U_x^{(i),n}}{|U_x^{(i),n-1}|}\cdot\varphi_{hx}^{(i)}dx\right)=0.$$

(Conincides with the scheme in [Barrett, Garcke, Nürnberg NMPDE 2011] if $\varepsilon = 0$.)

Numerical assessment

Condition number of the 'mass matrix' $\sim \epsilon^{-1}$:

l	$\epsilon_l = 0.3^{l-1}$	$\lambda_{\mathsf{max}}(\epsilon_l)$	$\lambda_{\mathsf{min}}(\epsilon_l)$	$\mathrm{cond}_2(\epsilon_l)$	$\mathrm{eoc}_{l-1,l}$
1	1	2.0025	0.33758	5.9	–
2	0.3	2.5482	0.14957	17.0	-0.8763
3	0.09	2.8415	0.050742	56.0	-0.9884
4	0.027	2.9451	0.016172	182.1	-0.9795
5	0.0081	2.9787	0.0051151	582.3	-0.9655
6	0.00243	2.9894	0.0016401	1822.7	-0.9478
7	0.000729	2.9928	0.00054014	5540.8	-0.9234
8	0.0002187	2.9939	0.00018427	16247.0	-0.8935
9	6.561e-05	2.9952	6.4619e-05	46351.0	-0.8707
10	1.9683e-05	2.9964	2.1764e-05	137680.0	-0.9042
11	5.9049e-06	2.9968	6.8319e-06	438640.0	-0.9624

Numerical assessment

Convergence test,
numerical reference solution,
$\epsilon = 10^{-3}$, $\delta = 0.2h^2$.

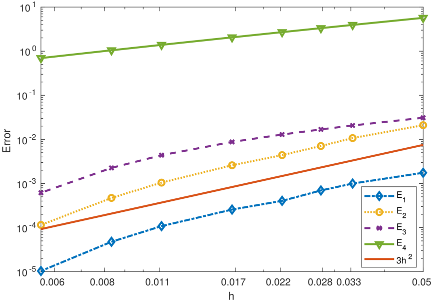
$$E_1 \simeq \|u - u_h\|^2_{L^\infty(L^\infty)},$$

$$E_2 \simeq \|u_x - u_{hx}\|^2_{L^\infty(L^2)}.$$

$$E_3 \simeq \|u_t - u_{ht}\|^2_{L^2(L^2)},$$

$$E_4 \simeq \max|\text{angle at junction} - 120^\circ|$$

test case with computed reference



Numerical assessment

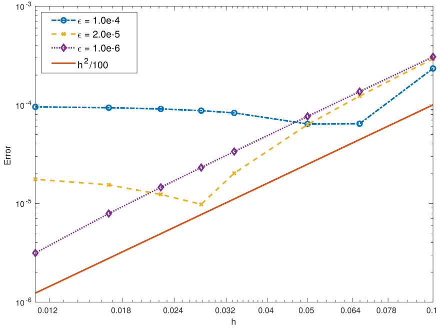
Convergence test and ε dependence,
analytical self-similar solution ($\varepsilon = 0$).

Slip BC not covered by theory!

Error for the red curve (distance):

$$\mathcal{E}_{curve}(J, \epsilon) := \max_{1 \le j \le J} \min_{x \in [0,1]} |U_j^{(1),N}(\epsilon) - u^{(1)}(x, T)|.$$

Varying h for several ε fixed:



test case with self-similar evolution

Varying ε for $h = 1/36$ fixed:

ϵ	$\mathcal{E}_{curve}$	eoc
1	0.62596	–
0.1	0.092471	0.8305
0.01	0.0097886	0.9753
0.001	9.7477e-04	1.0018
0.0001	8.7309e-05	1.0478
1e-05	1.6871e-05	0.7139

Numerical assessment

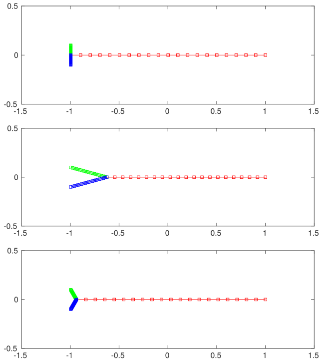
ϵ impact on the angle? Recall that

$$0 = \sum_{i=1}^{3} \left(1 + \epsilon |u_x^{(i)}|\right) \tau^{(i)}$$

Error of angles and triple junction position:

$$\mathcal{E}_{ang}(\epsilon) := \max_{1 \le i \le 3} |\theta_h^{(i)}(\epsilon) - 120|,$$

$$\mathcal{E}_{pos}(\epsilon) := |p_h(\epsilon) - p(0)|.$$



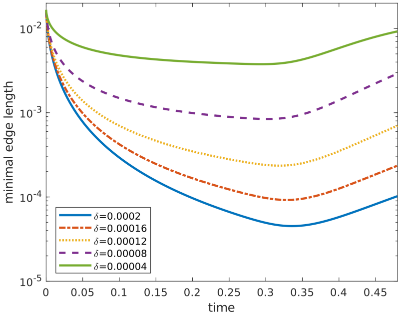
Spatial and time discretisation fixed:

J	N_{tot}	ϵ	$\mathcal{E}_{ang}$	eoc_{ang}	$\mathcal{E}_{pos}$	eoc_{pos}
20	669	1	89.719	–	0.31184	–
20	552	0.1	12.759	0.8471	0.015937	1.2915
20	3769	0.01	1.2665	1.0032	0.0014832	1.0312
20	18912	0.001	0.12656	1.0003	0.00014736	1.0028
20	8864	0.0001	0.012655	1.0000	1.4726e-05	1.0003
20	21	1e-05	0.001264	1.0006	1.4684e-06	1.0012

Further examples

Time step size?

Mesh quality:



spiral

Further examples

Self-intersection, 'jumping' a singularity.

self-intersection

Outline

Geometrically evolving triods minimising the network length

Relaxing the elastic energy of a closed curve

Elastic flow with length penalisation

Energy:

$$\mathcal{E}_{\tilde{\lambda}}(u) = \mathcal{E}(u) + \tilde{\lambda}\mathcal{L}(u) = \frac{1}{2}\int_0^{2\pi} |\kappa|^2 |u_x| dx + \tilde{\lambda}\int_0^{2\pi} |u_x| dx.$$

L^2 gradient flow (also in higher codimension):

$$u_t = -\nabla_s^2 \kappa - \frac{1}{2}|\kappa|^2\kappa + \tilde{\lambda}\kappa$$

Numerous analytical studies
[Langer, Singer 1985], [Koiso 1992],
[Wen 1993, 1995], [Polden 1996], [Mantegazza, Pluda, Pozzetta 2021],
and numerical studies
[Dziuk, Kuwert, Schätzle 2002], [Deckelnick, Dziuk 2009],
[Barrett, Garcke, Nürnberg 2007, 2010, 2012], [Balzani, Rumpf 2012],
[Bartels 2013], [Pozzi 2015], [Bondavara 2015].

Elastic flow with Dirichlet energy penalisation

Alternative: elastic energy with Dirichlet energy penalisation,

$$\mathcal{D}_\lambda(u) = \mathcal{E}(u) + \lambda \mathcal{D}(u) = \frac{1}{2}\int_0^{2\pi} |\kappa|^2 ds + \frac{1}{2}\lambda \int_0^{2\pi} |u_x|^2 dx.$$

L^2 gradient flow:

$$\begin{aligned} u_t &= -\nabla_s^2 \kappa - \frac{1}{2}|\kappa|^2 \kappa + \lambda \frac{u_{xx}}{|u_x|} \\ &= -\nabla_s^2 \kappa - \frac{1}{2}|\kappa|^2 \kappa + \lambda \kappa |u_x| + \lambda (|u_x|)_s \tau, \end{aligned}$$

involves tangential movements beneficial for the mesh quality (gradient flow, but no geometric flow).

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L^2 gradient flow:

$$\begin{aligned} u_t &= -\nabla_s^2\kappa - \frac{1}{2}|\kappa|^2\kappa + \lambda\frac{u_{xx}}{|u_x|} \\ &= -\nabla_s^2\kappa - \frac{1}{2}|\kappa|^2\kappa + \lambda\kappa|u_x| + \lambda(|u_x|)_s\tau, \end{aligned}$$

involves tangential movements beneficial for the mesh quality
(gradient flow, but no geometric flow).

Growth still is penalised but extremal points are the 'same':

If u is critical for $\mathcal{D}_\lambda$ then $|u_x|$ is constant
(consider variations in tangential direction).

Therefore, u is critical for $\mathcal{E}_{\tilde{\lambda}}$ with $\tilde{\lambda} = \lambda|u_x|$.

Finite element approximation

FE scheme and analysis following [Deckelnick, Dziuk 2009]:

Weak formulation:

$$\int_0^{2\pi} (u_t \cdot \phi)|u_x| - \int_0^{2\pi} \frac{P\kappa_x \cdot \phi_x}{|u_x|} - \frac{1}{2}\int_0^{2\pi} |\kappa|^2(\tau \cdot \phi_x) + \lambda \int_0^{2\pi} u_x \cdot \phi_x = 0$$

$$\int_0^{2\pi} (\kappa \cdot \psi)|u_x| + \int_0^{2\pi} (\tau \cdot \psi_x) = 0$$

$$u(0, \cdot) = u_0$$

with $P = I - \tau \otimes \tau$ projection to normal space.

Assume that there is a unique smooth, periodic (in space) solution, which is regular:

$$c_0 \le |u_x| \le C_0, \qquad |\kappa| \le C_0.$$

Finite element approximation

FE scheme and analysis following [Deckelnick, Dziuk 2009]:

Semi-discrete problem (linear finite elements in space):

$$\int_0^{2\pi} I_h(u_{ht} \cdot \phi_h)|u_{hx}| - \int_0^{2\pi} \frac{P_h \kappa_{hx} \cdot \phi_{hx}}{|u_{hx}|} - \frac{1}{2}\int_0^{2\pi} I_h(|\kappa_h|^2)(\tau_h \cdot \phi_{hx}) + \lambda \int_0^{2\pi} u_{hx} \cdot \phi_{hx} = 0$$

$$\int_0^{2\pi} I_h(\kappa_h \cdot \psi_h)|u_{hx}| + \int_0^{2\pi} (\tau_h \cdot \psi_{hx}) = 0$$

$$u_h(0, \cdot) = I_h u_0$$

with $P_h = I - \tau_h \otimes \tau_h$, I_h interpolation operator.

Natural energy identity preserved:

$$\int_0^{2\pi} I_h(|u_{ht}|^2)|u_{hx}| + \frac{d}{dt}\left\{\frac{1}{2}\int_0^{2\pi} I_h(|\kappa_h|^2)|u_{hx}| + \frac{\lambda}{2}\int_0^{2\pi} |u_{hx}|^2\right\} = 0.$$

Theorem

Convergence and error estimate [Pozzi, S, ESAIM M2AN 2023]

For all h small enough the semi-discrete problem has a unique solution and is such that

$$\sup_{t\in[0,T]} \|u(t,\cdot) - u_h(t,\cdot)\|_{H^1}^2 + \int_0^T \|u_t(t,\cdot) - u_{ht}(t,\cdot)\|_{L^2}^2 dt \leq Ch^2, \tag{1}$$

$$\sup_{t\in[0,T]} \|\kappa(t,\cdot) - \kappa_h(t,\cdot)\|_{L^2}^2 + \int_0^T \|\kappa_x(t,\cdot) - \kappa_{hx}(t,\cdot)\|_{L^2}^2 dt \leq Ch^2 \tag{2}$$

Proof follows the lines of [Deckelnick, Dziuk 2009]

1. Short time well-posedness.
2. Error estimates (several technical lemmas),
   need to control $|u_{hx}|$.
3. Full time interval for h small enough.

Numerical assessment

Simple time discretisation (linear saddle point):

$$\int_0^{2\pi} I_h\Big(\frac{u_h^{(m+1)} - u_h^{(m)}}{\delta} \cdot \phi_h\Big)|u_{hx}^{(m)}| - \int_0^{2\pi} \frac{P_h^{(m)} \kappa_{hx}^{(m+1)} \cdot \phi_{hx}}{|u_{hx}^{(m)}|}$$

$$-\frac{1}{2}\int_0^{2\pi} I_h\big(|\kappa_h^{(m)}|^2\big)\Big(\frac{u_{hx}^{(m+1)}}{|u_{hx}^{(m)}|} \cdot \phi_{hx}\Big) + \lambda \int_0^{2\pi} u_{hx}^{(m+1)} \cdot \phi_{hx} = 0,$$

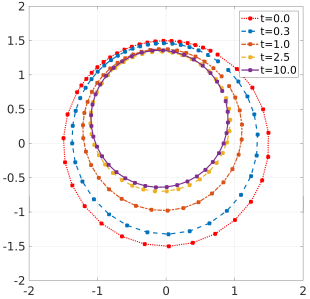
$$\int_0^{2\pi} I_h\big(\kappa_h^{(m+1)} \cdot \psi_h\big)|u_{hx}^{(m)}| + \int_0^{2\pi} \Big(\frac{u_{hx}^{(m+1)}}{|u_{hx}^{(m)}|} \cdot \psi_{hx}\Big) = 0,$$

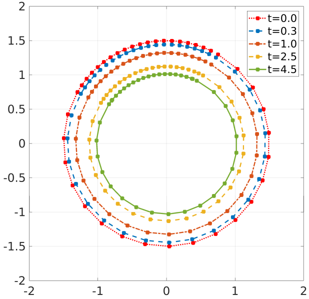
Numerical assessment

Radially symmetric solution, initially equidistributed mesh points, $\mathrm{err} = |\kappa - \kappa_h|^2$:

N	h	m_T	δ	err	eoc
20	0.31416	400	0.0025	1.556e-05	–
30	0.20944	900	0.0011111	3.0805e-06	3.9944
36	0.17453	1296	0.0007716	1.4864e-06	3.997
46	0.13659	2116	0.00047259	5.5786e-07	3.998
60	0.10472	3600	0.00027778	1.9279e-07	3.9988

Initially non-equidistributed mesh points (flow not geometric!)

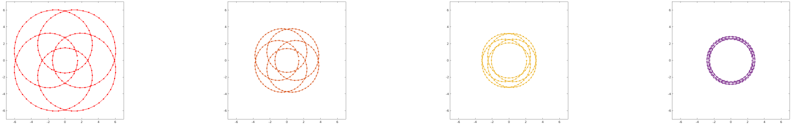




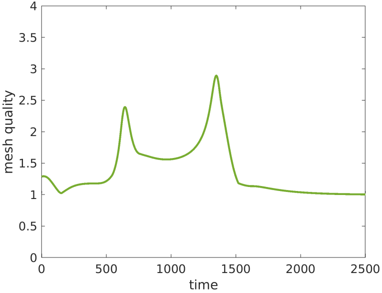
Scheme from [Deckelnick, Dziuk 2009].

Numerical assessment

Hypocycloid, 2D: [Barrett, Garcke, Nürnberg 2007+]



3D, slight off-plane perturbation:
[Deckelnick, Dziuk 2007]



hypocycloid

Extensions

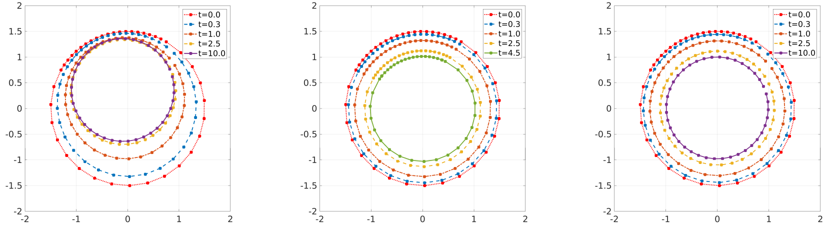
Idea from networks, normal movement $\sim \varepsilon$, keep tangential movement. Energy:

$$\mathcal{E}(u) + \tilde{\lambda}\mathcal{L}(u) + \epsilon\mathcal{D}(u),$$

weighted L^2 gradient flow:

$$Pu_t + \varepsilon(u_t \cdot \tau)\tau = -\nabla_s^2\kappa - \frac{1}{2}|\kappa|^2\kappa + \tilde{\lambda}\kappa + \varepsilon\Big(\kappa|u_x| + \lambda(|u_x|)_s\tau\Big).$$

Initially non-equidistributed mesh points:



left: original scheme,
middle: scheme from [Deckelnick, Dziuk 2009],
right: ε weighted scheme (geometric up to ε error).

(New scheme / preprint [Deckelnick, Nürnberg 2024] without ε error.)

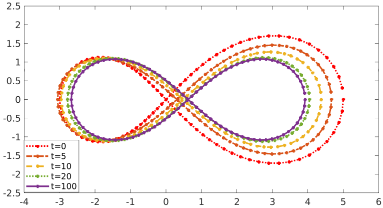
Extensions

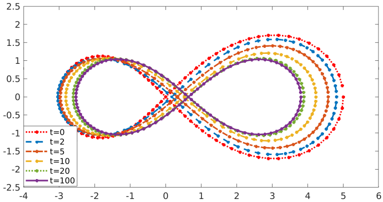
Ideas from [Mackenzie, Nolan, Rowlatt, Insall 2019]:
Monitoring function M, weighting in Dirichlet term,

$$\int_0^{2\pi} u_x \cdot \phi_x dx \quad \longrightarrow \quad \int_0^{2\pi} M(u,\kappa,x) u_x \cdot \phi_x dx.$$

The higher M the higher the 'tension' $\rightsquigarrow$ vertices move closer.

Relaxation of a non-symmetric lemniscate:





left: scheme in [Barrett, Garcke, Nürnberg 2007] with $\tilde{\lambda} = 0.2$,
right: ε and M weighted scheme, same $\tilde{\lambda}$,

$$M = M(x_1) = 1 + \frac{(x_1 - 1)^2}{10} \quad \rightsquigarrow \text{ more mesh points away from centre.}$$

Conclusion

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Collaborator: Paola Pozzi (U Duisburg-Essen, Germany).

Thanks for your attention!

Finite element approximation

Weak formulation:

$$\begin{aligned}\mathcal{T}_P := \{\, \Gamma = (u^{(1)}, u^{(2)}, u^{(3)}) \,|\, u^{(i)} \in W^{1,2}(\Omega, \mathbb{R}^2) \text{ regular a. e.}, \\ u^{(i)}(1) = P_i, \quad i = 1,2,3, \\ u^{(1)}(0) = u^{(2)}(0) = u^{(3)}(0)\,\}.\end{aligned}$$

Find $\Gamma(t) = (u^{(1)}(t), u^{(2)}(t), u^{(3)}(t)) \in \mathcal{T}_P$, $t \in [0, T]$, such that $\forall \varphi \in \mathcal{T}_0$

$$\begin{aligned}\sum_{i=1}^{3} \Bigg(\int_\Omega (u_t^{(i)} \cdot \frac{(u_x^{(i)})^\perp}{|u_x^{(i)}|})(\varphi^{(i)} \cdot \frac{(u_x^{(i)})^\perp}{|u_x^{(i)}|})|u_x^{(i)}|dx \\ + \epsilon \int_\Omega (u_t^{(i)} \cdot \frac{u_x^{(i)}}{|u_x^{(i)}|})(\varphi^{(i)} \cdot \frac{u_x^{(i)}}{|u_x^{(i)}|})|u_x^{(i)}|^2 dx \Bigg) \\ = -\sum_{i=1}^{3} \Bigg(\epsilon \int_\Omega u_x^{(i)} \cdot \varphi_x^{(i)} dx + \int_0^1 \frac{u_x^{(i)}}{|u_x^{(i)}|} \cdot \varphi_x^{(i)} dx \Bigg).\end{aligned}$$

Finite element approximation

Semi-discrete problem: (using piecewise linear FEs, space S_h)

$$\mathcal{T}_{P,h} := \{\,\Gamma_h = (u_h^{(1)}, u_h^{(2)}, u_h^{(3)}) \mid u_h^{(i)} \in S_h^2 \text{ regular a. e.}, \quad u_h^{(i)}(1) = P_i, \quad i = 1,2,3, \quad u_h^{(1)}(0) = u_h^{(2)}(0) = u_h^{(3)}(0)\,\},$$

Find $\Gamma_h(t) = (u_h^{(1)}(t), u_h^{(2)}(t), u_h^{(3)}(t)) \in \mathcal{T}_{P,h}$, $t \in [0,T]$, such that $\forall \varphi_h \in \mathcal{T}_{0,h}$

$$\sum_{i=1}^{3} \Bigg(\int_\Omega (u_{ht}^{(i)} \cdot \frac{(u_{hx}^{(i)})^\perp}{|u_{hx}^{(i)}|})(\varphi_h^{(i)} \cdot \frac{(u_{hx}^{(i)})^\perp}{|u_{hx}^{(i)}|})|u_{hx}^{(i)}|dx + \epsilon \int_\Omega (u_{ht}^{(i)} \cdot \frac{u_{hx}^{(i)}}{|u_{hx}^{(i)}|})(\varphi_h^{(i)} \cdot \frac{u_{hx}^{(i)}}{|u_{hx}^{(i)}|})|u_{hx}^{(i)}|^2 dx \Bigg) = -\sum_{i=1}^{3} \Bigg(\epsilon \int_\Omega u_{hx}^{(i)} \cdot \varphi_{hx}^{(i)} dx + \int_\Omega \frac{u_{hx}^{(i)}}{|u_{hx}^{(i)}|} \cdot \varphi_{hx}^{(i)} dx \Bigg).$$

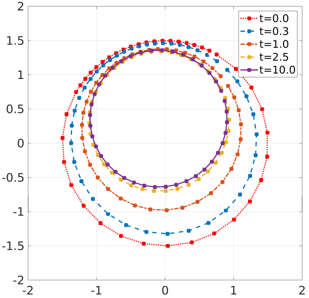
Numerical assessment

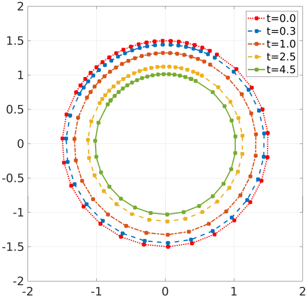
Numerical assessment

Radially symmetric solution, initially equidistributed mesh points, $\mathrm{err} = |y - y_h|^2$:

N	h	m_T	δ	err	eoc
20	0.31416	400	0.0025	1.556e-05	–
30	0.20944	900	0.0011111	3.0805e-06	3.9944
36	0.17453	1296	0.0007716	1.4864e-06	3.997
46	0.13659	2116	0.00047259	5.5786e-07	3.998
60	0.10472	3600	0.00027778	1.9279e-07	3.9988

Initially non-equidistributed mesh points:

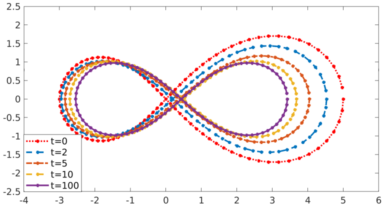


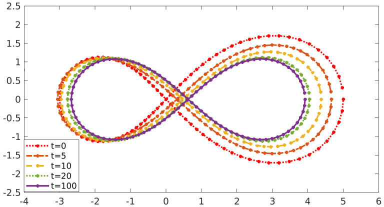


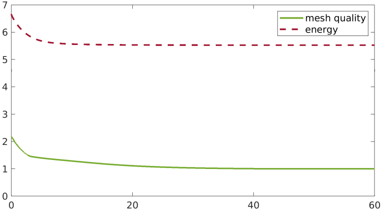
Right: scheme from [Deckelnick, Dziuk 2009].

Numerical assessment

Relaxation of a non-symmetric lemniscate:



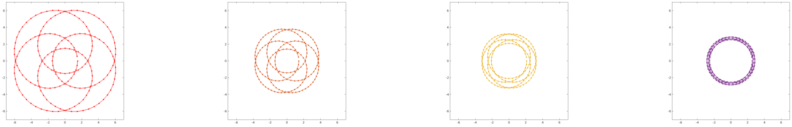




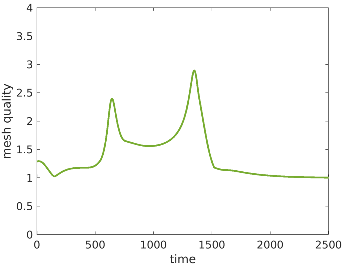
Top left: new methods with $\lambda = 0.1$.
Top right: scheme in [Barrett, Garcke, Nürnberg 2007] with $\tilde{\lambda} = 0.2$.

Numerical assessment

Hypocycloid, 2D:



3D, slight off-plane perturbation:



hypocycloid