# Numerical surgery for mean curvature flow of surfaces

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*"What happens when we let a numerical analyst to the operating table?"*

Balázs Kovács
Paderborn University

The 81st Fujihara Seminar

Mathematical Aspects for Interfaces and Free Boundaries

6 June 2024, Niseko, Japan

# Outline

- **Motivation**
- Theoretical results on MCF with surgery
- A convergent algorithm for MCF
- Surgery
  - Continuous surgery
  - Numerical surgery
- Numerical experiments

## Ethical statement

- None of the test subjects were *permanently* harmed during the presented test-operations.
- Viewer discretion is not advised.

# Patient – Mean curvature flow

Surface evolution under mean curvature flow:

$$v = -H\nu,$$

where $H$ is the mean curvature and $\nu$ is the outward unit normal field of $\Gamma[X]$.

The flow has nice properties (e.g. maximum principle, avoidance property, etc.). [Huisken (1984)], [...]

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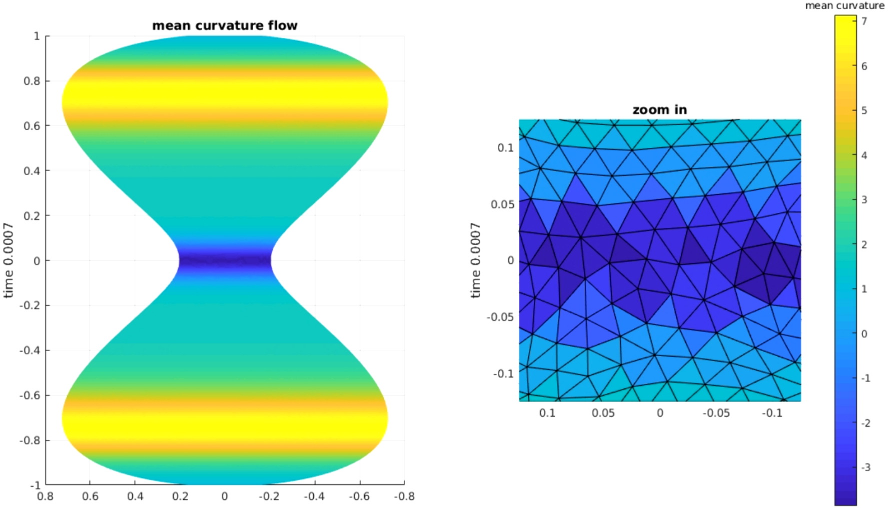
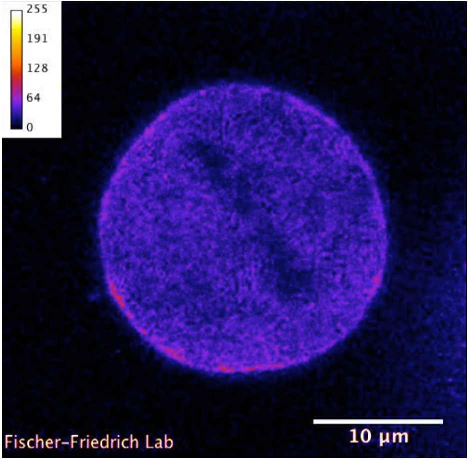
The "heat equation" for surface flows.
**Singularities occur** for $d \geq 2$.

# A flow requiring "treatment" – a pinch singularity



# A flow requiring "treatment" – a pinch singularity

# In **reality** – cell division by contractile ring formation




Experiment by E. Fischer-Friedrich (TU Dresden & FOR 3013).
[Wittwer and Aland (2022)], [Bonati, Wittwer, Aland, and Fischer-Friedrich (2022)]

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# The Cure – Goal of this talk

**How can we pass through singularities?**

**We do surgery!**

**How can we pass through singularities for *parametric methods*?**

**We do surgery!**

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Ideas are from Ricci flow with surgery:

- Hamilton (1982,1993,1997) $\longleftarrow$ Hamilton's programme
- Perelman (2002,2003a,b) $\longrightarrow$ solving the Poincaré conjecture

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# Further literature

Mean curvature flow with surgery

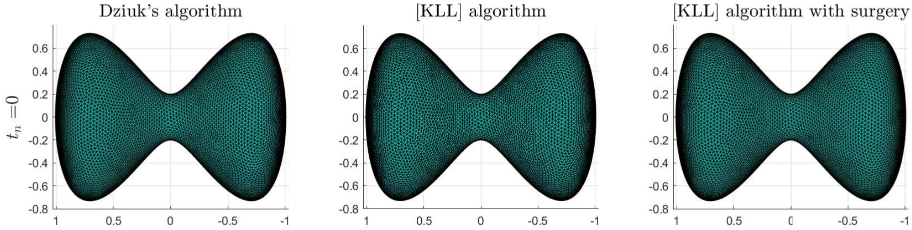
- Huisken and Sinestrari (2009)
- Brendle and Huisken (2016,2018)
- etc.

Numerical approaches

- just compute beyond a singularity (e.g. [Dziuk (1990)])
- Balazovjech, Mikula, Petrášová, and Urbán (2012)
- Benninghoff and Garcke (2014,2016a,b,2017)

  a global approach based on a background grid
- non-parametric methods (level-set, phase-field, XFEMs, etc.)

# Surgery in action



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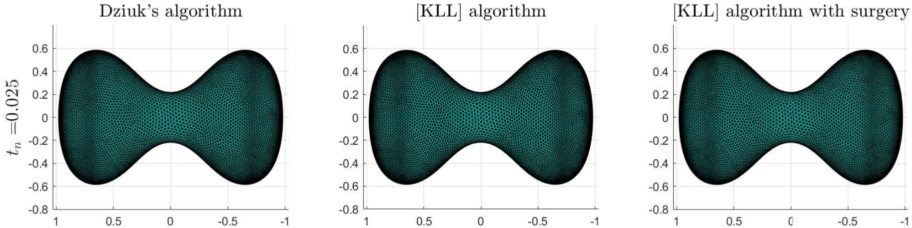

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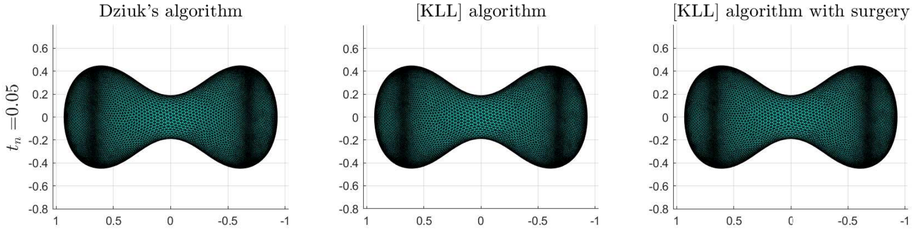

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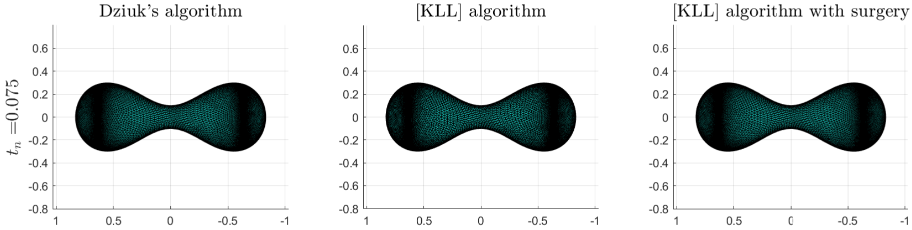

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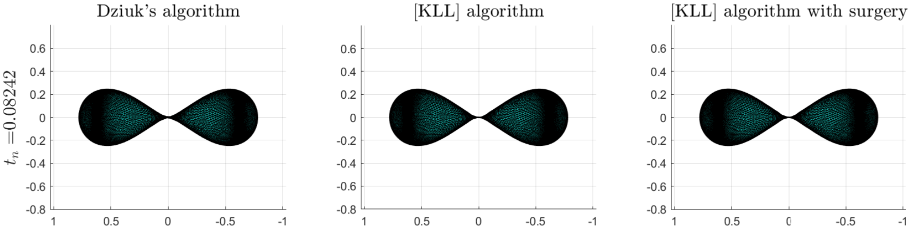

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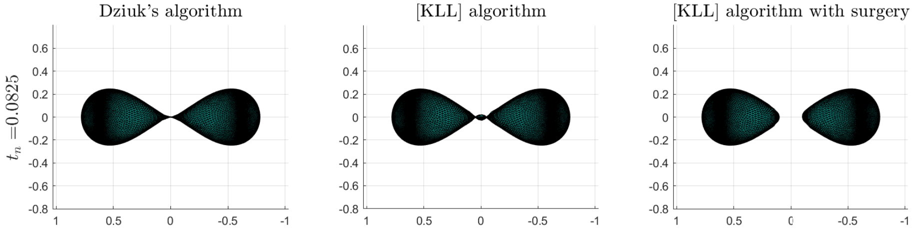

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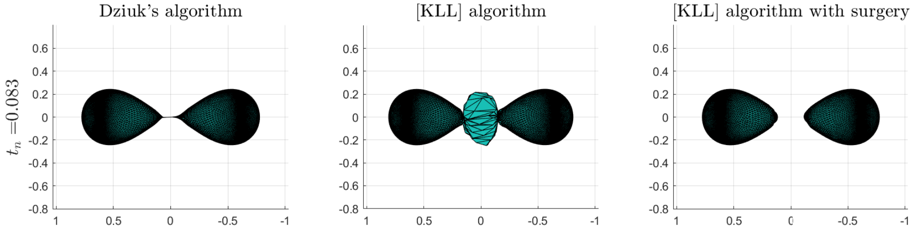

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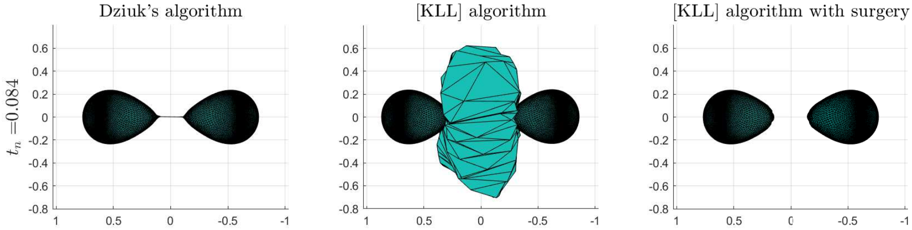

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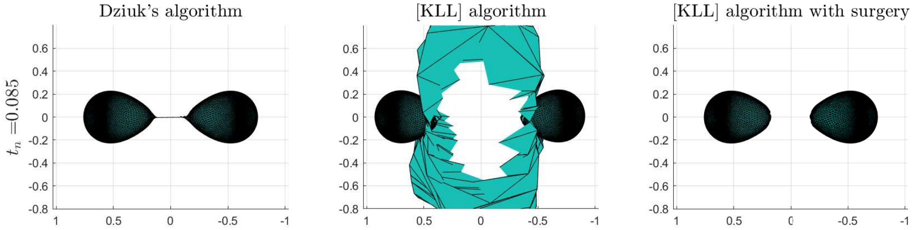

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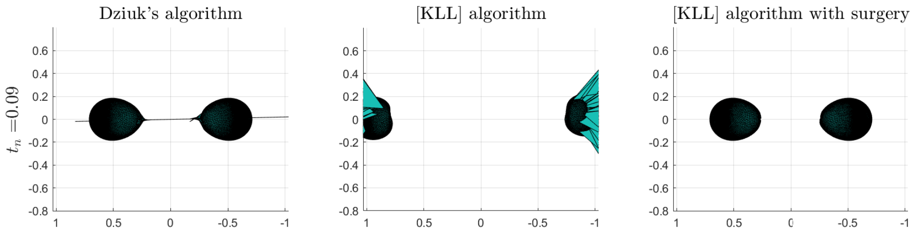

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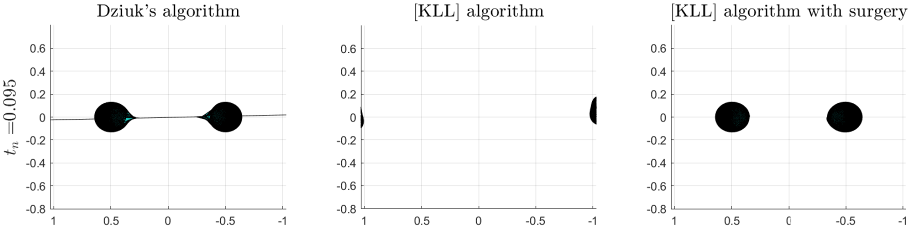

# Mean curvature flow with surgery

# The three keys

The Huisken–Sinestrari-type MCF with surgery relies on:

- convexity estimate, i.e. regions of high curvature have almost positive definite second fundamental form;
- **cylindrical estimate**, i.e. for a 2-convex flow, regions of high curvature are either uniformly convex or close to a cylinder;
- gradient estimate, which gives derivative bounds depending only at the curvature at a single point.

[Haslhofer and Kleiner (2017)]

# Mean curvature flow with surgery – I.

## Algorithm (Mean curvature flow with surgery )

***Data:*** *Let $\Gamma^0 \subset \mathbb{R}^{n+1}$ be a closed embedded initial surface.*
*Let the curvature thresholds $H_1 < H_2 < H_3$ be given.*

*(a) Let the smooth flow evolve $\Gamma(t)$ until time $t$ such that $\max\{H(\cdot,t)\} > H_3$.*

*(b) Perform surgeries on necks, removing all points with curvature greater than $H_2$.*
*Right after surgery, maximal curvature drops below $H_2$.*

*(c) Repeat the Steps (a) and (b) until the flow goes extinct.*

[Huisken and Sinestrari (2009)] ($n \geq 3$, $\kappa_1^0 + \kappa_2^0 \geq 0$)
[Brendle and Huisken (2016)] ($\mathbb{R}^3$, $H^0 > 0$)

# Mean curvature flow with surgery – II.

**Theorem** [Brendle and Huisken (2016)]

Let $\Gamma^0$ be a closed, embedded surface in $\mathbb{R}^3$, with positive mean curvature. Then there exists a mean curvature flow with surgeries starting from $\Gamma^0$ which terminates after finitely many steps.

**Challenges**:

- Step (b) "perform surgeries on necks": how? scaling?
  key technique: $\alpha$-noncollapsedness ($R_{\text{in}}(x) \geq \alpha / H(x)$)
  key issue: $\max\{H\} \leq H_2$ except on the regions diffeomorphic to a sphere, or a "neck"
- send the curvature thresholds $H_1 < H_2 < H_3$ to infinity, the flow with surgery converges to the level set solution

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# A convergent algorithm for mean curvature flow with numerical surgery

# The analysts' approach

A regular surface $\Gamma[X]$ moving under mean curvature flow satisfies the coupled system:

$$v = -H\nu,$$

$$\partial^\bullet \nu = \Delta_{\Gamma[X]}\nu + |\nabla_{\Gamma[X]}\nu|^2\,\nu,$$
$$\partial^\bullet H = \Delta_{\Gamma[X]}H + |\nabla_{\Gamma[X]}\nu|^2 H,$$

$$\partial_t X = v.$$

[Huisken (1984)]

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**[K., Li and Lubich (2019)]:**
**Numerical algorithm based on this coupled system.**

# Discrete mean curvature flow with surgery

## Algorithm (Fully discrete mean curvature flow with surgery)

***Data:*** *Let $\Gamma_h^0$ be the discretised initial surface, with curvature $H_h^0$ and normal vector field $\nu_h^0$.*
*Let the curvature thresholds $H_3 > H_2$ be given.*

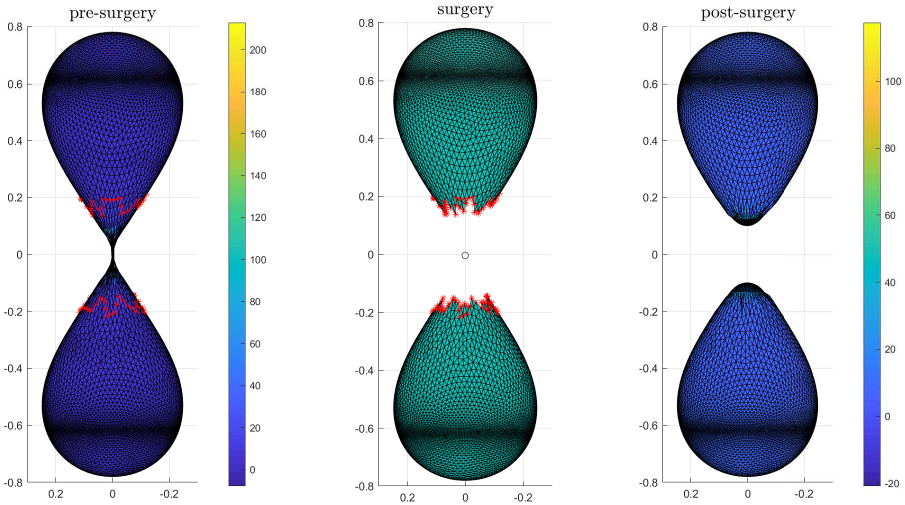
*(a) Let the fully discrete algorithm [KLL] evolve $\Gamma_h[\mathbf{x}^n]$, $H_h^n$, and $\nu_h^n$ until time $t_n$ such that $\max\{H_h^n\} > H_3$.*

*(b) Perform surgeries on $\Gamma_h[\mathbf{x}^n]$, removing all nodes for which $H_h^n(\mathbf{x}_j^n) > H_2$.*
*Right after surgery maximal curvature drops to $H_2$.*

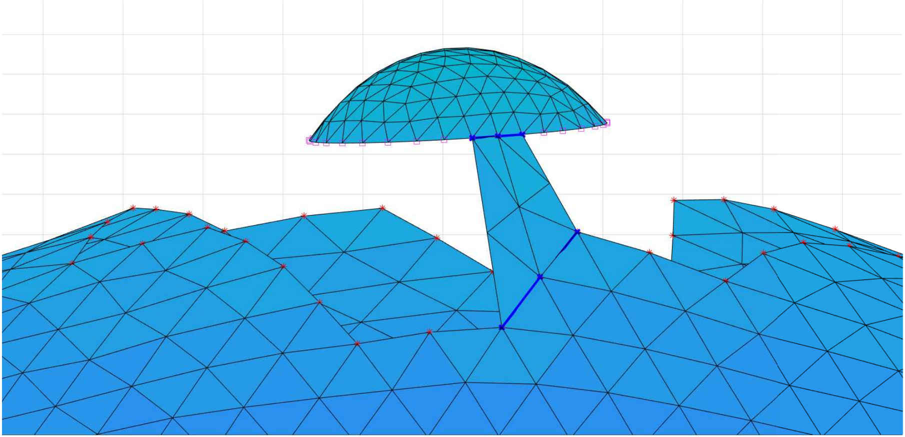
*(c) Repeat the Steps (a) and (b) until the discrete flow goes extinct.*

Step (b) is technical (removing elements, computing new geometry, etc.).

# Step (b) – sketch



# Step (b) – sewing



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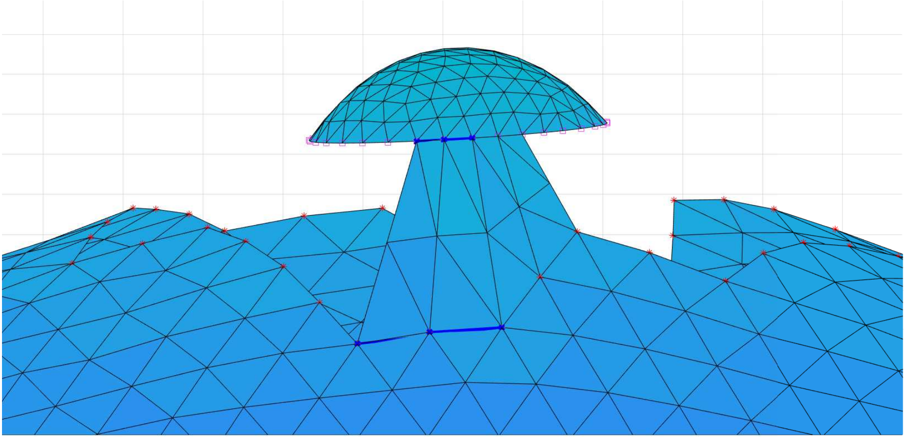

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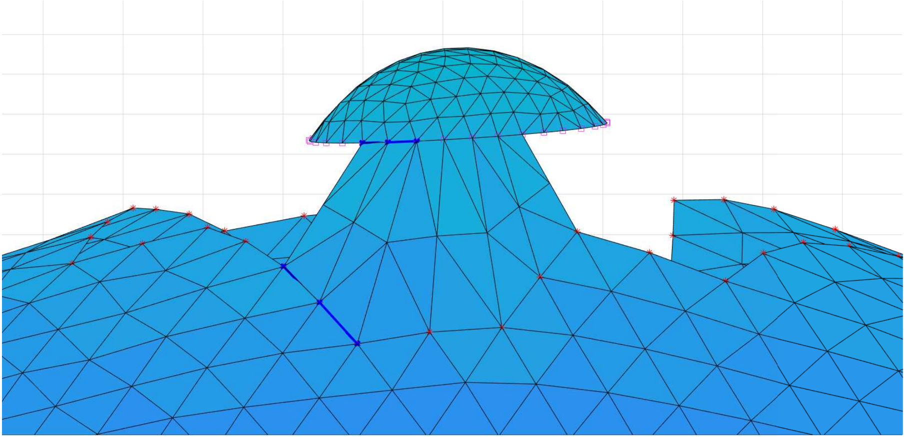

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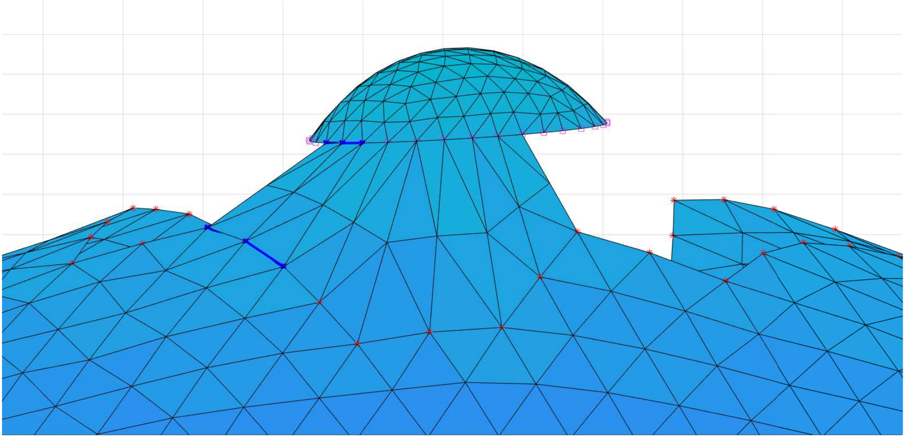

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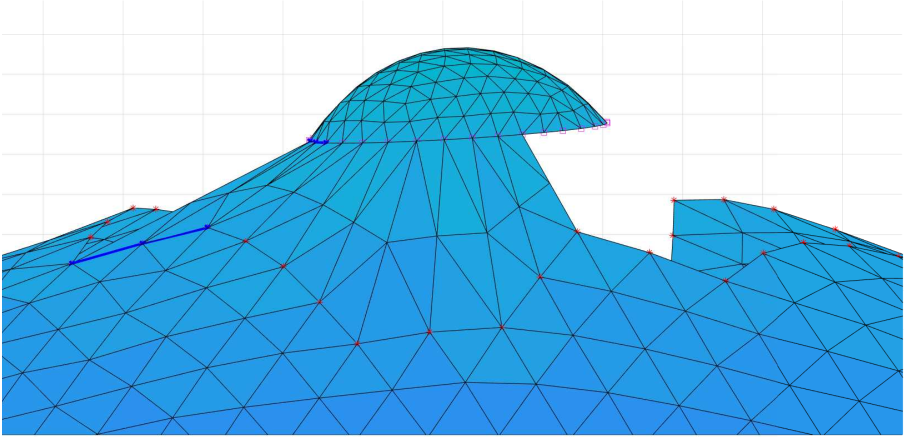

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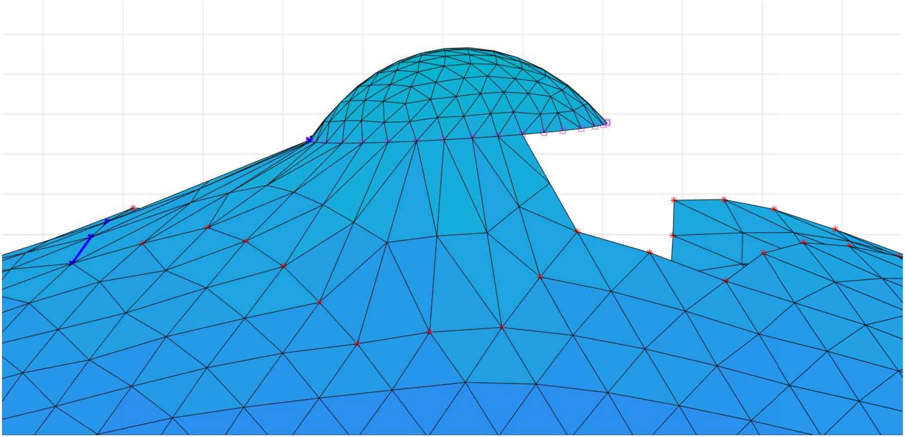

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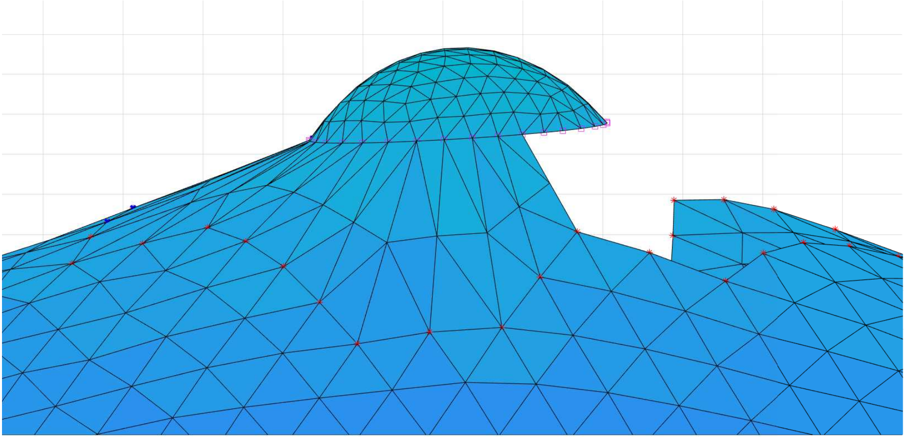

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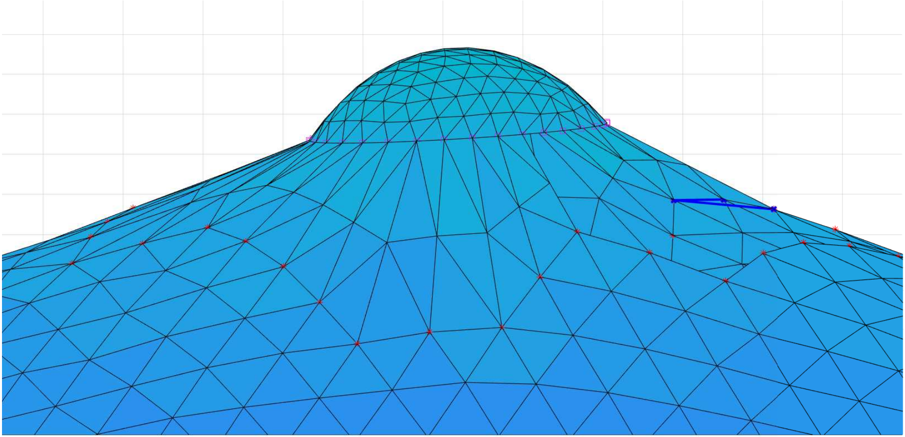

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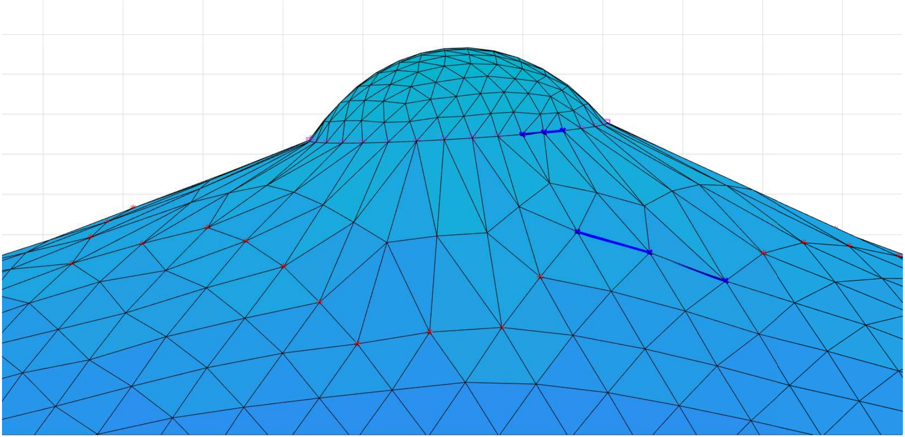
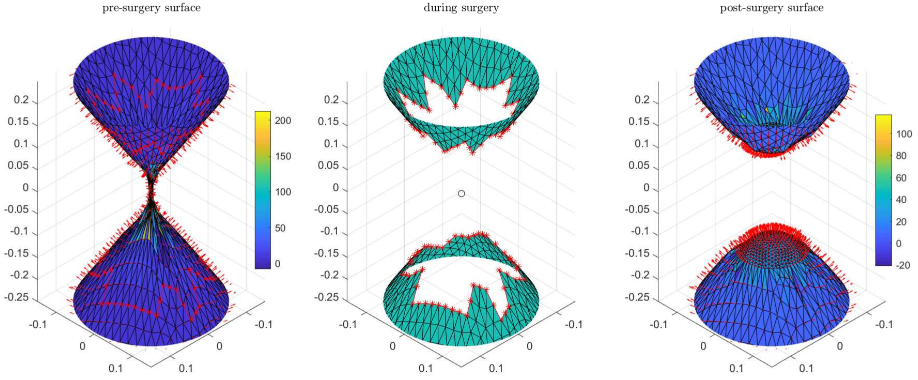
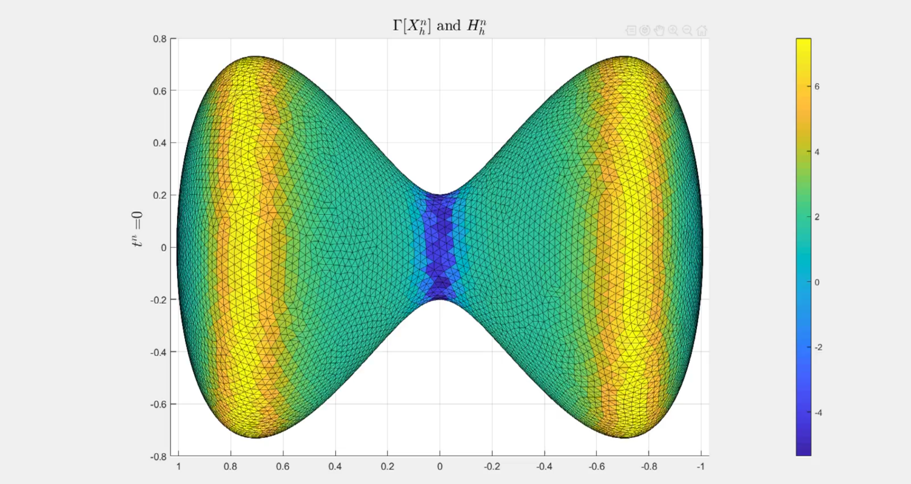


# Step (b) – sketch (zoom-in)



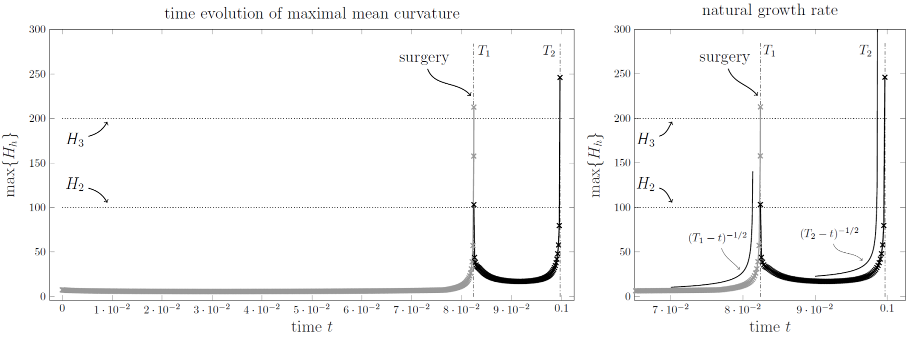
# Numerical experiments

# Test-surgery I.

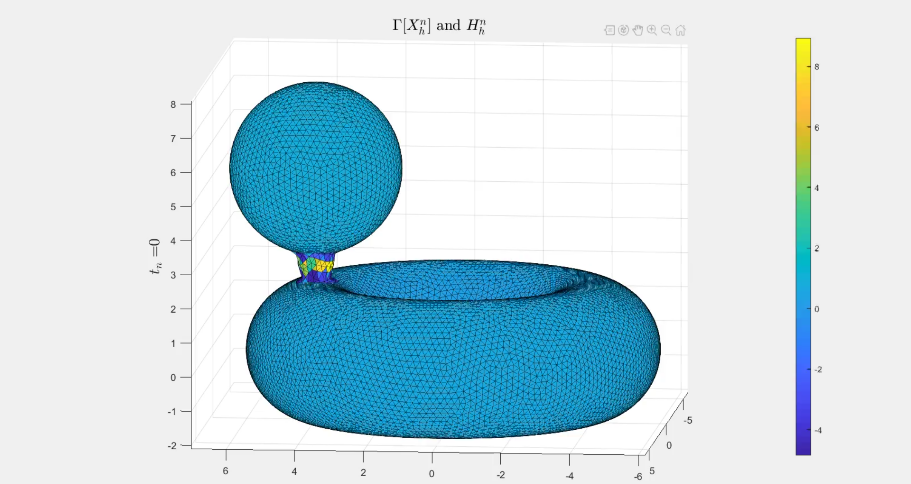


# Test-surgery I.

# Time evolution and scaling of $\max\{H_h^n\}$



# Test-surgery II.



# Test-surgery II.

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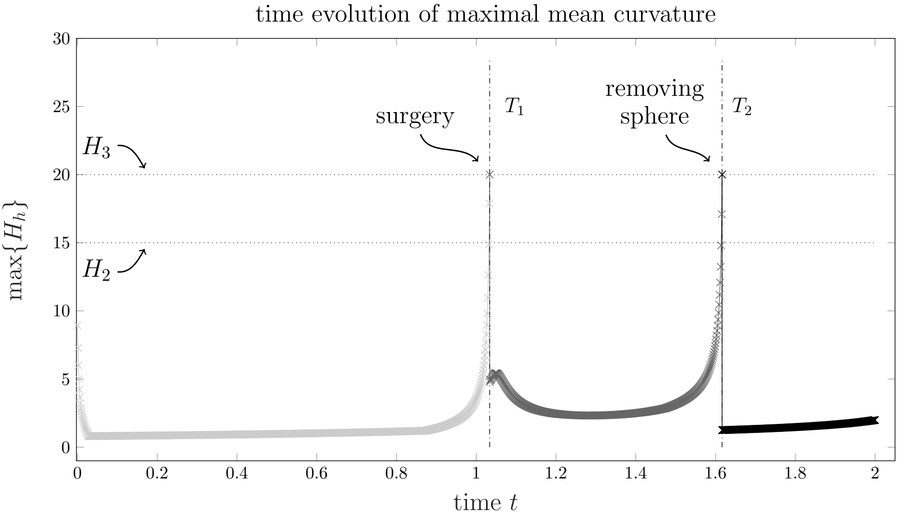

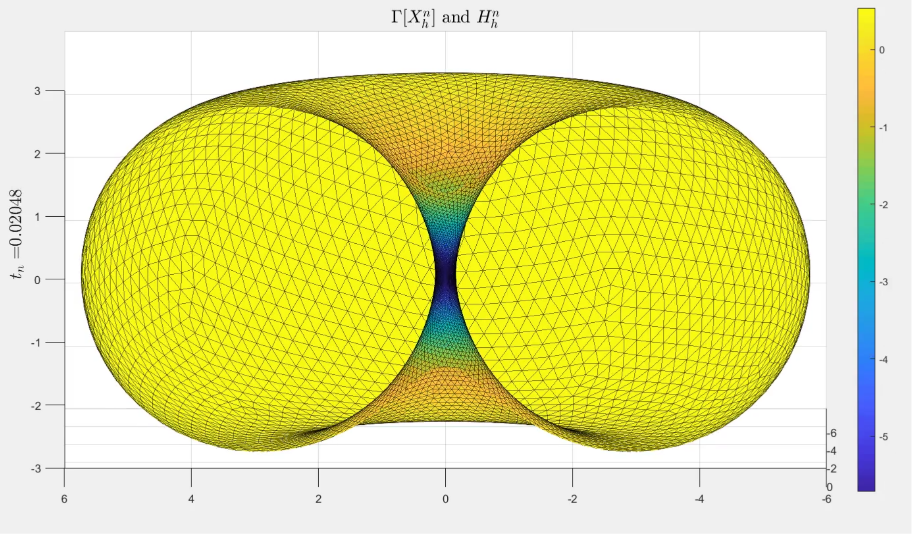

$\Gamma[X_h^n]$ and $H_h^n$
$t_n = 0.02048$
3
2
1
0
-1
-2
-3
6
4
2
0
-2
-4
-6
-6
-4
-2
0
0
-1
-2
-3
-4
-5

**Thank you for your attention!**