On a new narrow band level set method

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Joint work with Maxim Olshanskii (Houston), Paul Schwering (RWTH)

Level set method in analysis

Evolution of initial surface $\Gamma_0 \subset \mathbb{R}^3$ that evolves with normal velocity V_N?

Use $\phi(\cdot, t) : \mathbb{R}^3 \to \mathbb{R}$ with level sets $\Gamma_c(t) := \{\, x \mid \phi(x, t) = c \,\}$.

Initialization: $\phi(\cdot, 0) \sim$ signed distance to Γ_0.

Key principle:
all level sets Γ_c of ϕ evolve with their normal velocity $V_N = V_N(\Gamma_c)$.

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Example: mean curvature flow

$$\text{Normal velocity} \quad V_N = -\text{mean curvature} = -\kappa$$

Using $\kappa = \mathrm{div}(\mathbf{n}) = \mathrm{div}(\frac{\nabla\phi}{|\nabla\phi|})$ one obtains

Level set PDE for mean curvature flow

$$\frac{\partial\phi}{\partial t} - |\nabla\phi|\mathrm{div}(\frac{\nabla\phi}{|\nabla\phi|}) = 0 \quad \text{in} \;\; \Omega \subset \mathbb{R}^3, \;\; t \geq 0$$

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The notion of viscosity solutions fits well: There exists a unique global-in-time viscosity solution ϕ for suitable $\phi(\cdot, 0)$.

Weak notion can handle singularities.

Implict surface evolution: $\Gamma_0(t) = \{\, \phi(\cdot, t) = 0 \,\}$ ("fattening")

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Extensive work in literature on analysis of PDEs:

[Chen, Giga, Evans, Spruck, 1991]

[Y. Giga, Surface Evolution Equations: A Level Set Approach (2006)]

[X. Bian, Y. Giga, and H. Mitake, A level-set method for a mean curvature flow with a prescribed boundary, Preprint (2023)]

......many more......

Level set method in numerics

Many applications, cf. [//math.berkeley.edu/sethian/], talk of J. Sethian

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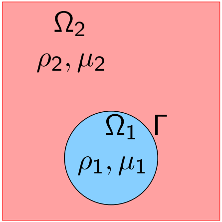
One example: Two-phase incompressible flows (cf. talk of H. Garcke)

Interface: $\Gamma(t) = \partial\Omega_1 \cap \partial\Omega_2$

$\mathbf{D}(\mathbf{u}) = \nabla\mathbf{u} + \nabla\mathbf{u}^T$, $\sigma = -p\,\mathbf{I} + \mu\mathbf{D}(\mathbf{u})$

κ: curvature

τ: surface tension coefficient



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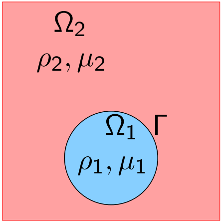
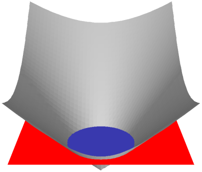


Coupled Navier-Stokes equations

$$\begin{cases} \rho_i(\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u}) = -\nabla p + \operatorname{div}(\mu_i \mathbf{D}(\mathbf{u})) + \rho_i \mathbf{g} & \text{in } \Omega_i \\ \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega_i \end{cases} \quad \text{for } i = 1, 2$$

$$[\sigma\mathbf{n}]_\Gamma = \tau\kappa\mathbf{n} \quad \text{(surface tension)}, \quad [\mathbf{u}]_\Gamma = 0\,, \;\; V_N = \mathbf{u}\cdot\mathbf{n}.$$

$\Gamma(t)$ = zero-level of $\varphi(x,t)$



$$\varphi(x,t) = \begin{cases} < 0 & \text{for } x \text{ in phase } \color{blue}{\Omega_1} \\ > 0 & \text{for } x \text{ in phase } \color{red}{\Omega_2} \\ = 0 & \text{at the interface} \end{cases}$$

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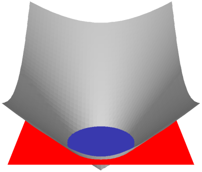

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Navier-Stokes equations coupled with level set equation

$$\begin{aligned} \rho(\varphi)\Big(\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u}\Big) - \operatorname{div}\Big(\mu(\varphi)\,\mathbf{D}(\mathbf{u})\Big) + \nabla p &= \rho(\varphi)\,g - \tau\,\kappa(\varphi)\,\delta_\Gamma \mathbf{n}_\Gamma \\ \nabla\cdot\mathbf{u} &= 0 \\ \varphi_t + \mathbf{u}\cdot\nabla\varphi &= 0 \end{aligned}$$

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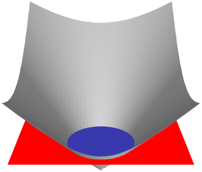

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Method can deal with droplet merging/splitting.

Note: $\mathbf{u}$ globally defined.

Narrow band level set method: motivation

An example (our motivation): **surface** Navier-Stokes equations:

$$\begin{cases} \rho\dot{\mathbf{u}} = -\nabla_\Gamma p + 2\mu\,\mathrm{div}_\Gamma(E_s(\mathbf{u})) + \mathbf{b} + p\kappa\mathbf{n} \\ \mathrm{div}_\Gamma \mathbf{u} = 0 \end{cases} \qquad \text{on } \Gamma(t),$$

Geometric evolution of $\Gamma(t)$ is defined by

$$\mathbf{u}_N = (\mathbf{u}\cdot\mathbf{n})\mathbf{n}, \quad \partial_t X = \mathbf{u}_N \circ X; \qquad X(\cdot,t) : \text{parametrization of } \Gamma(t)$$

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Other example: Mean curvature coupled with surface diffusion, cf.

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In numerics: Lagrangian or Eulerian approach.

Lagrangian approach: cf. talks of C. Elliott, H. Garcke.

Eulerian approach $\rightsquigarrow$ narrow band level set method

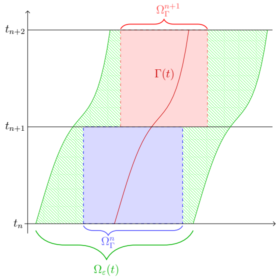
Narrow band level set method

Assumption:
$\mathbf{u}_N$ extended to neighbourhood of $\Gamma(t)$.

Evolving narrow band:

$\Omega_\epsilon(t) := \{ x \in \mathbb{R}^d \mid |\phi(x,t)| < \epsilon \}$

Time stepping on narrow band Ω_Γ^n.



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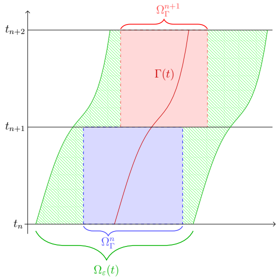

Structure of narrow band algorithm

a) Given ϕ_h^n specify boundary conditions ϕ_D^n on inflow boundary of Ω_Γ^n.

b) Given ϕ_h^n and boundary data ϕ_D^n: solve LS equation approximately on $\Omega_\Gamma^n \times [t_n, t_{n+1}]$ (we use DG FEM). Result $\tilde{\phi}_h^{n+1}$.

c) Find ϕ_h^{n+1} as an extension of $\tilde{\phi}_h^{n+1}$ from Ω_Γ^n to Ω_Γ^{n+1}.

$\Rightarrow$ an extension procedure is needed.

Extension techniques

Known techniques: re-initialization

- Fast marching method (FMM, [Sethian, 1996])
- PDE based: solve Eikonal equation locally
- .....

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use the Ghost penalty (GP) technique [Burman et al.]

GP: has several applications as stabilization technique in FEM .

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GP: has several applications as stabilization technique in FEM .

We propose to use it as extension method.

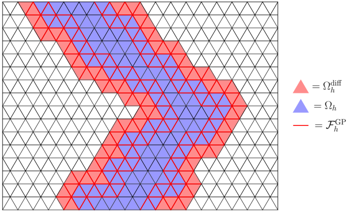
Ghost penalty based extension

Given Ω_h and extended domain Ω_h^{ex}.

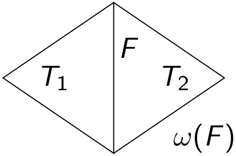
GP faces: $\mathcal{F}_h^{\mathsf{GP}}$

$\omega(F) = T_1 \cup T_2$ for $F \in \mathcal{F}_h^{\mathsf{GP}}$

V_h: continuous FE of degree k.



$\psi \in V_h$: $\psi_i = \mathcal{E}^P(\psi_{|T_i})$ (polynomial extension).



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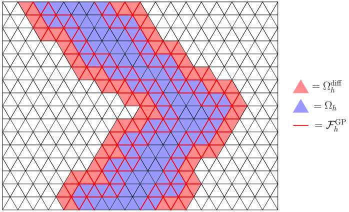

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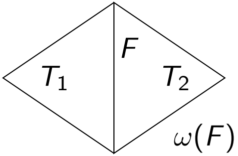

GP bilinear form ("volumetric jump" formulation [J. Preuss, 2018])

$$s_h(\phi, \psi) := \gamma \sum_{F \in \mathcal{F}_h^{\mathrm{GP}}} \int_{\omega(F)} (\phi_1 - \phi_2)(\psi_1 - \psi_2)\, dx, \quad \text{for } \phi, \psi \in V_h(\Omega_h^{\mathrm{ex}}),$$

with parameter $\gamma > 0$.

Ghost penalty based extension

$(\phi, \psi)_\omega := (\phi, \psi)_{L^2(\omega)}$

GP extension bilinear form

For a given function $\tilde{\phi} \in L^2(\Omega_h)$, determine $\phi_h \in V_h(\Omega_h^{\mathrm{ex}})$ such that
$a_h^{\mathrm{ext}}(\phi_h, \psi_h) := \boldsymbol{(\phi_h, \psi_h)_{\Omega_h} + s_h(\phi_h, \psi_h) = (\tilde{\phi}, \psi_h)_{\Omega_h}}$ for all $\psi_h \in V_h(\Omega_h^{\mathrm{ex}})$

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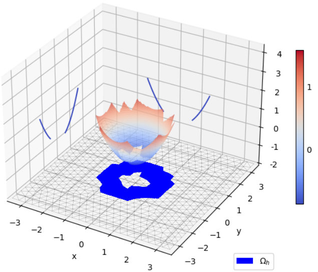

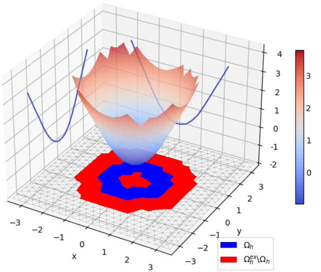

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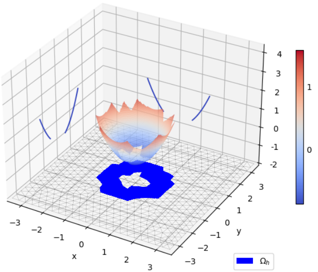

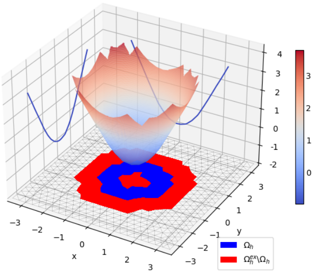

Note: instead of $(\phi_h, \psi_h)_{\Omega_h}$ also $(\phi_h, \psi_h)_{H^1(\Omega_h)}$ can be used.

Error analysis

Assumption: Ω_h may be of width $\sim h$; $\Omega_h^{\mathrm{ex}} \setminus \Omega_h$ must be of width $\sim h$.

We assume a $\phi \in H^{k+1}(\Omega_h^{\mathrm{ex}})$, and $\tilde{\phi} \approx \phi$ on Ω_h.

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Stability and error bounds

$$\|\psi_h\|_{\Omega_h^{\mathrm{ex}}}^2 \leq c\, a_h^{\mathrm{ext}}(\psi_h, \psi_h) \quad \text{for all} \quad \psi_h \in V_h(\Omega_h^{\mathrm{ex}})$$

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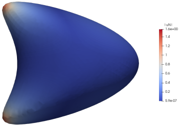
- General numerical extension operator
- Optimal error bound
- Still open: error bound for whole narrow band discretization method

Numerical experiment for extension method

"Kite" level set function $\phi(\mathbf{x}) = (x_1 - x_3^2)^2 + x_2^2 + x_3^2 - 1$. Zero level:



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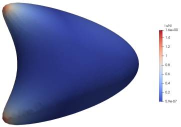


$\mathcal{T}_\Gamma$: tetrahedra cut by Γ_h (zero level)

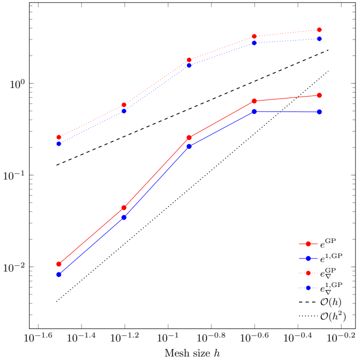
$\Omega_h := \mathcal{N}(\mathcal{N}(\mathcal{T}_\Gamma))$

$\Omega_h^{\rm ext} := \mathcal{N}(\mathcal{N}(\Omega_h))$ (two additional layers)

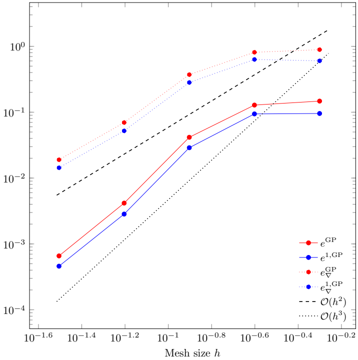
$\tilde{\phi} := (I_h \phi)_{|\Omega_h}$ (input for extension problem)

Error measures:

$$e^{\mathrm{GP}} = \frac{\|\phi - \phi_h\|_{\Omega_h^{\mathrm{ex}}}}{\|1\|_{\Omega_h^{\mathrm{ex}}}}, \qquad e_{\nabla}^{\mathrm{GP}} = \frac{\|\nabla(\phi - \phi_h)\|_{\Omega_h^{\mathrm{ex}}}}{\|1\|_{\Omega_h^{\mathrm{ex}}}}$$



$k = 1$

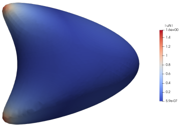


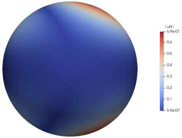
$k = 2$

$k = 4$: $\phi_h = \phi$.

Numerical experiment for narrow band level set method

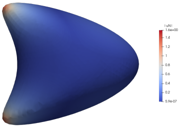
Kite to sphere level set function $\phi(x, t)$.

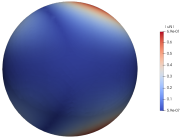




Numerical experiment for narrow band level set method

Kite to sphere level set function $\phi(x, t)$.





On $\Omega_h^n \times [t_n, t_{n+1}]$:
Inflow boundary data: $(\phi_h^n)_{|\text{boundary}}$

Spatial discretization: DG FEM, degree $k = 1$.

Time discretization: BDF2, $\Delta t \sim h$.

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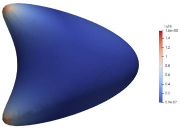

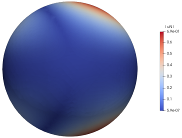

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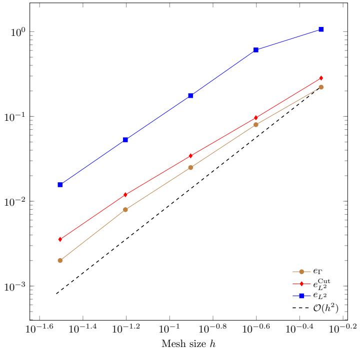
Spatial discretization: DG FEM, degree $k = 1$.

Time discretization: BDF2, $\Delta t \sim h$.

In extension: $k = 1$
$\Omega_{\mathrm{proj}} :=$ smallest set of T that contains $|\phi_h^n| \leq h$.
$\Omega_h^{\mathrm{ext}} := \Omega_h^{n+1}$.

Error measures:

$$e_\Gamma^2 = \Delta t \sum_{n=1}^{N} \frac{\|\phi(\cdot, t_n)\|^2_{\Gamma_h^n}}{\|1\|^2_{\Gamma_h^n}}, \qquad \left(e_{L^2}^{\mathrm{Cut}}\right)^2 = \Delta t \sum_{n=1}^{N} \frac{\|\phi - \phi_h^n\|^2_{\mathcal{T}_\Gamma}}{\|1\|^2_{\mathcal{T}_\Gamma}}$$



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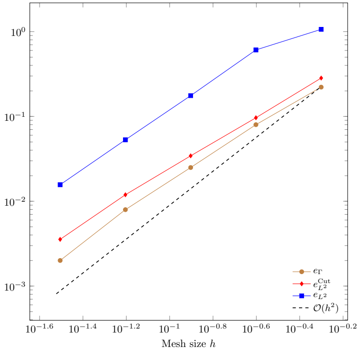

For (close to) singular or nonsmooth geometries: combination with robust re-initialization techniques.

Summarizing

- New extension method based on Ghost penalty technique
- Suitable for combination with FE discretization methods
- Error analysis is available. Higher order straightforward.
- Used as component in narrow band level set method
- Reference: preprint soon available in arXiv