Singular Neumann boundary problems for a class of fully nonlinear parabolic equations

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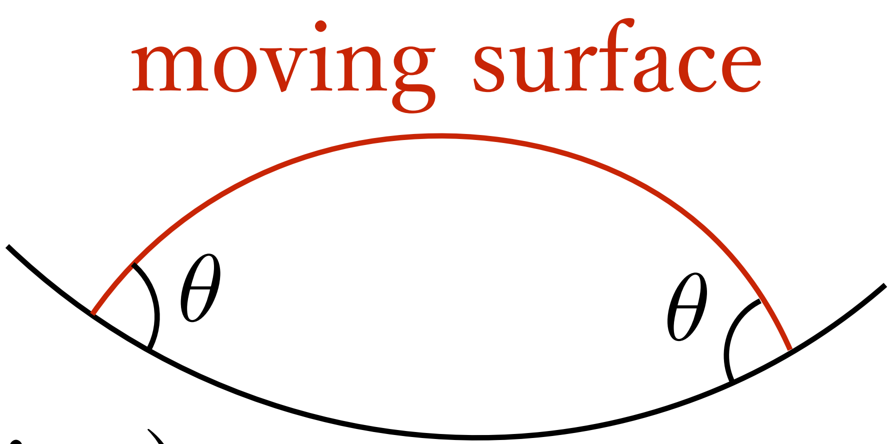
2022/6/9

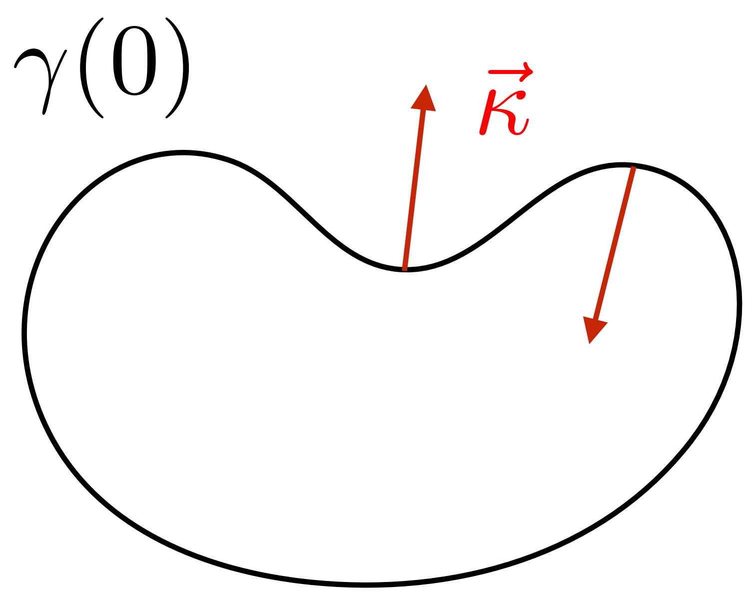
The 81st Fujihara Seminar
Mathematical Aspects for Interfaces and Free Boundaries

Introduction

- Main topic: Analysis on motion of hyper-surface generating contact angles on a “barrier” hyper-surface
- (Example of) physical backgrounds:
  - (i) Interface dynamics
  - (ii) Capillary problem (Effect of surface tension)



Ex of (i): • (Mean) curvature flow : $V = \kappa$ on $\gamma(t)$
(introduced to describe the motion of grain boundary in annealing by Mullins ’57)



V : normal velocity
κ : curvature
$n = 1$: dimension of curve (surface)

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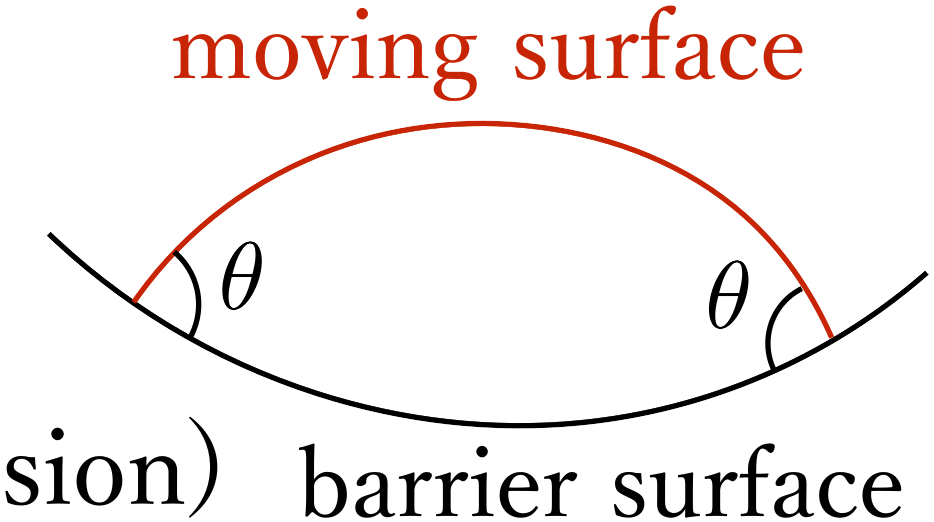

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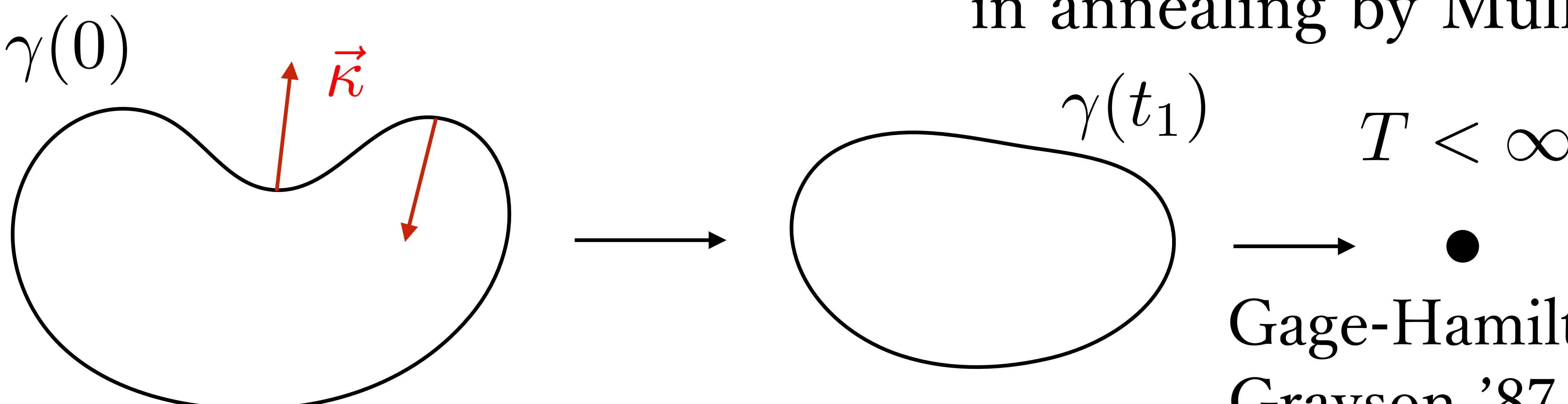

Gage-Hamilton '86
Grayson '87

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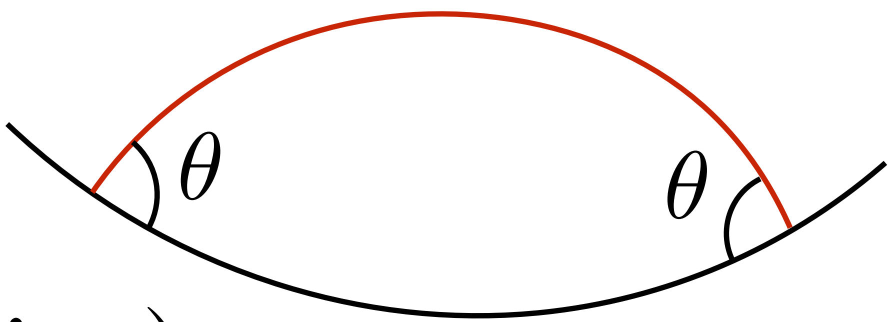

Ex of (i): • (Mean) curvature flow : $V = \kappa$ on $\gamma(t)$

• Anisotropic (mean) curvature flow :

$$V = \kappa_\phi \text{ on } \gamma(t)$$

(ϕ : surface energy density depending on normal velocity $\vec{\nu}$)

• Curve (surface) diffusion : $V = -\partial_s^2 \kappa$ on $\gamma(t)$

(s : arclength)

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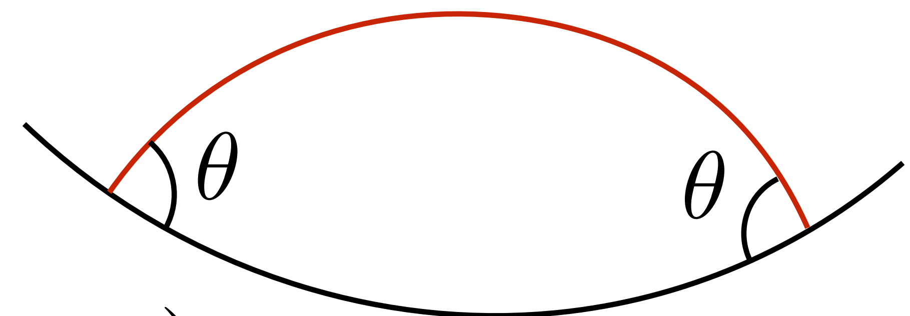
    (Effect of surface tension)

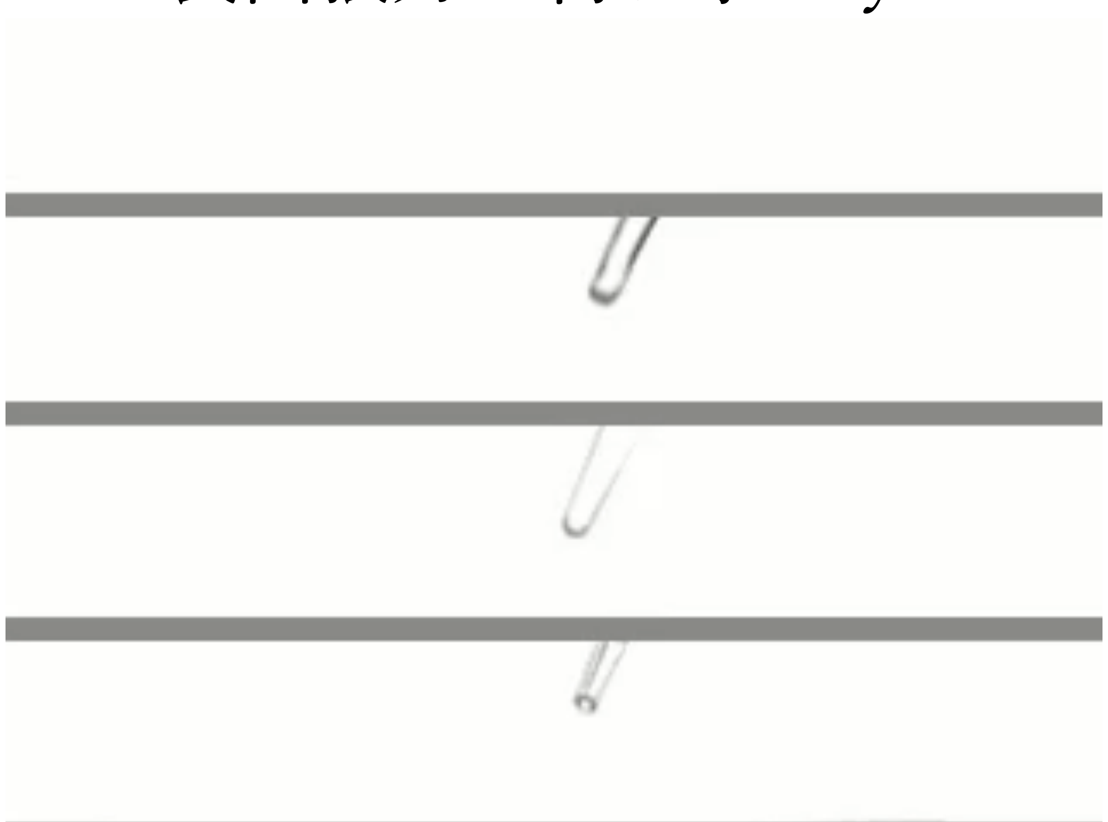
moving surface

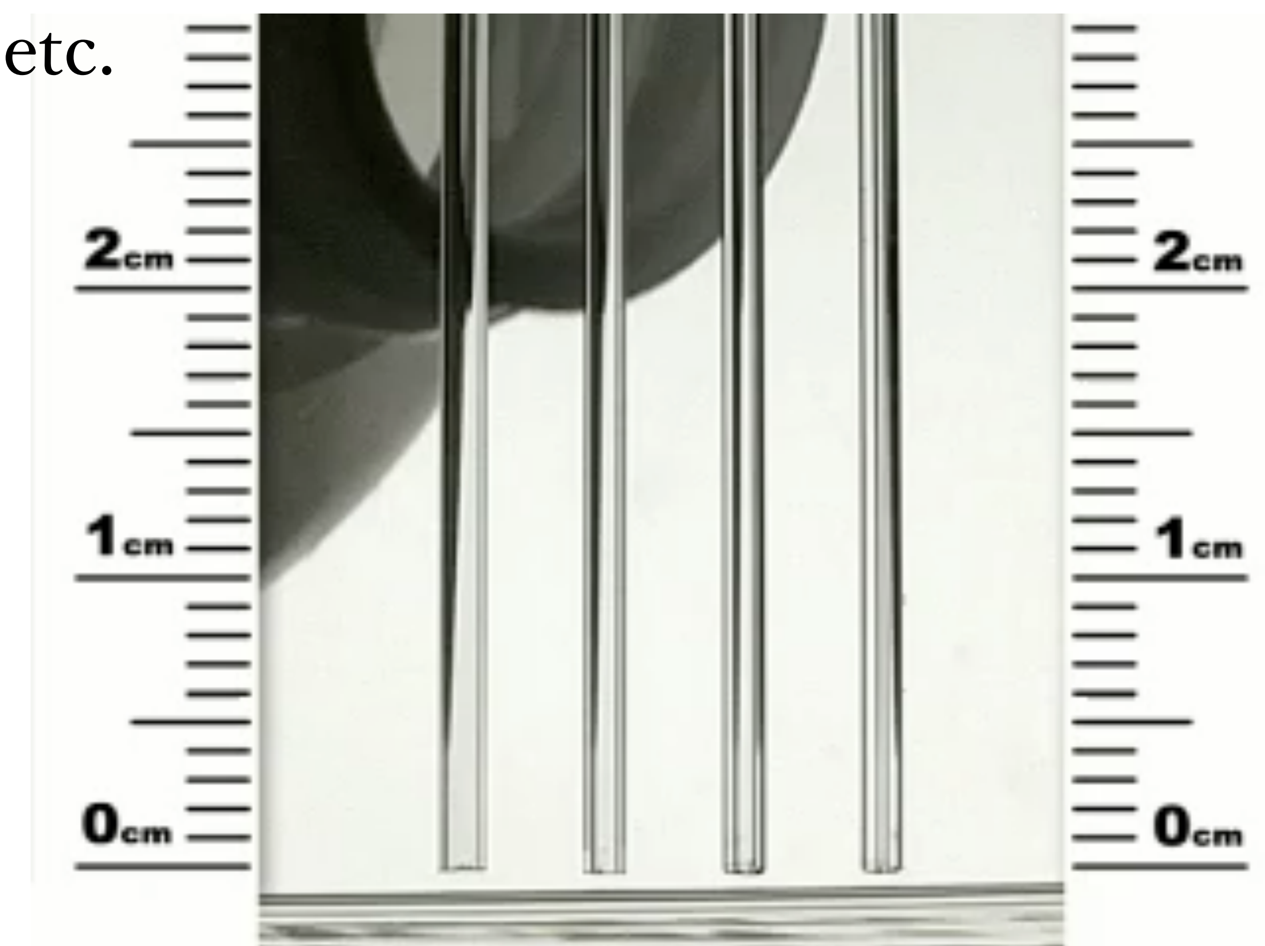


barrier surface

Movies for (ii):

from “表面張力の物理学” by de Gennes etc.





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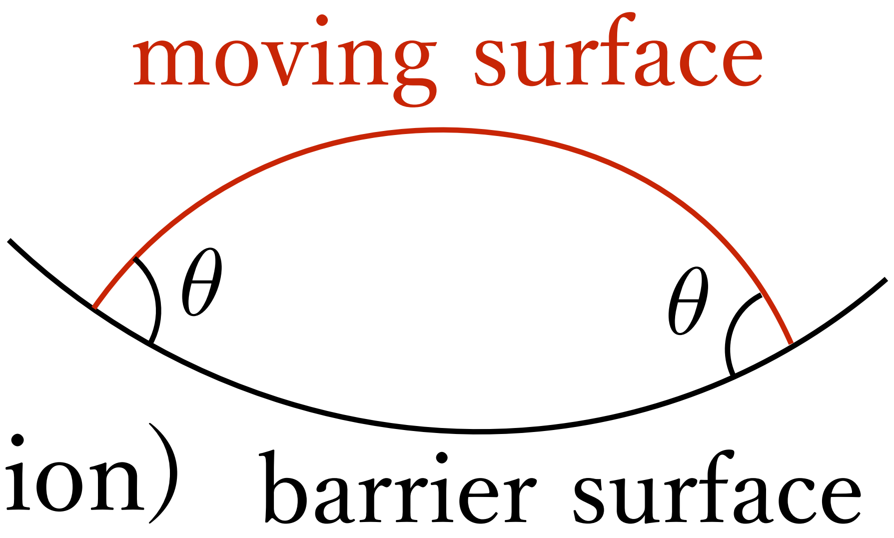


Motivation:

- Can we describe the driving force effect due to surface tension in mathematical model?
- Can we find new structure or behavior in some "imaginary" or "limiting" situations? (We will discuss tangentially contact case)

Known results for capillary model

- Mean curvature flow $(n \geq 2, \theta = \frac{\pi}{2})$

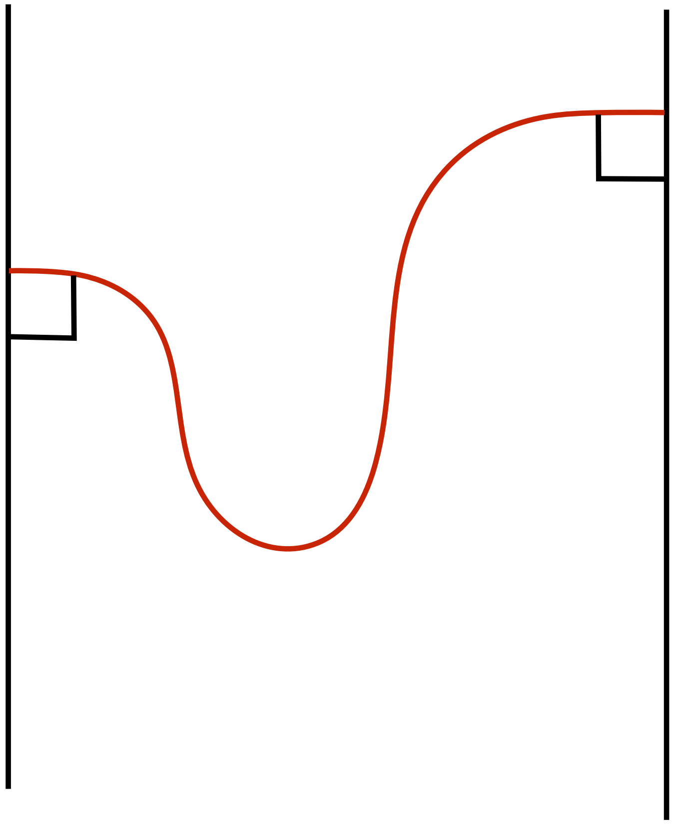
  Huisken '89

  - $\Omega \subset \mathbb{R}^n$: bounded domain
  - $u : \Omega \times [0, \infty) \to \mathbb{R}$

    : unknown function for

$$\begin{cases} \mathrm{graph}\, u(\cdot, t) \text{ moves by MCF} \\ \mathrm{graph}\, u(\cdot, t) \perp \partial\Omega \times \mathbb{R} \end{cases}$$

$\gamma(t) = \mathrm{graph}\, u(\cdot, t)$



✦ Energy structure: surface area decreases

✦ Maximum principle: $\sup_{x \in \Omega} |u(x,t)| \leq \sup_{x \in \Omega} |u(x,0)|$

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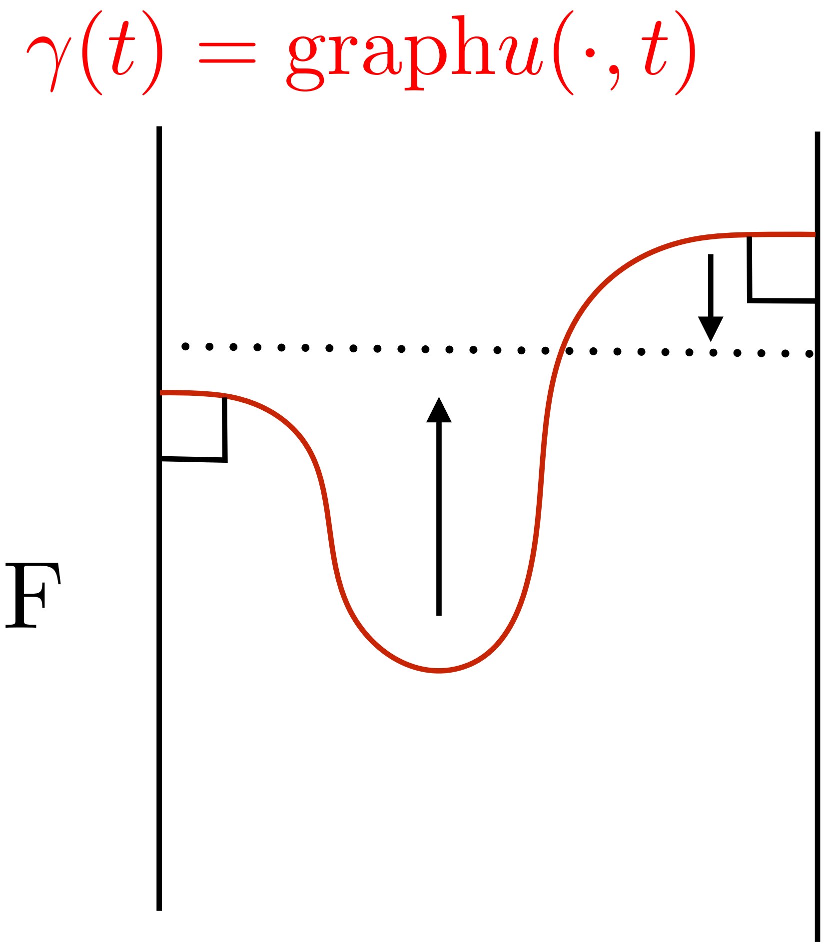


✦ Energy structure: surface area decreases

✦ Maximum principle: $\sup_{x\in\Omega} |u(x,t)| \leq \sup_{x\in\Omega} |u(x,0)|$

Results: $u(\cdot,0) \in C^{2,\alpha} \Rightarrow \exists! u$: global-in-time smooth sol.

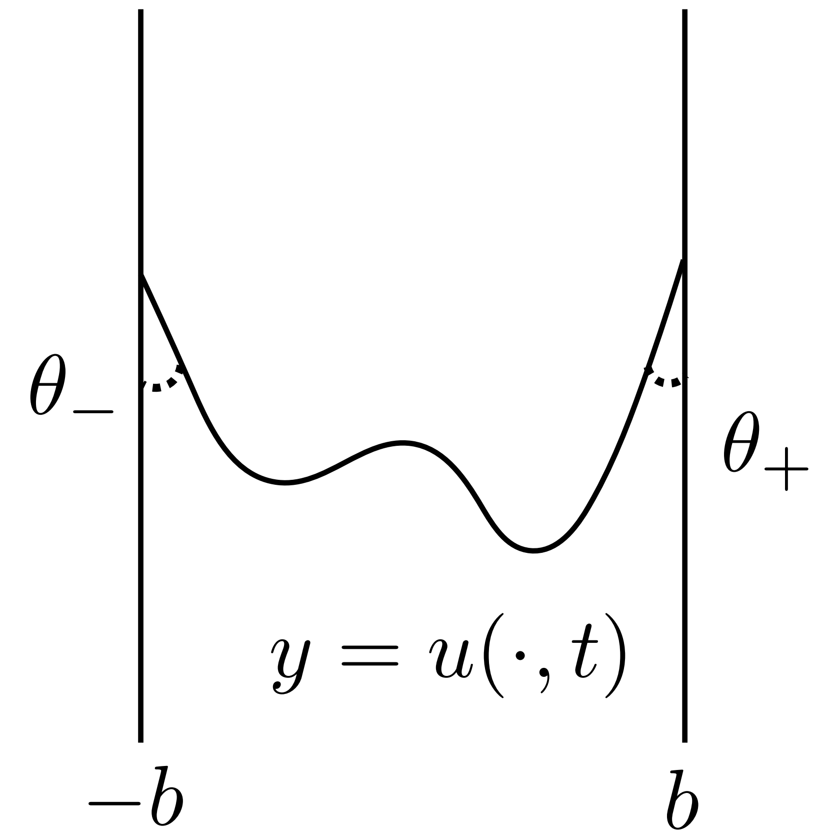
and it converges to constant as $t \to \infty$

Introduction

- Capillary rise model (curvature flow type $n = 1$)

$(I = (-b, b)$: interval, $g \in C^\infty(\mathbb{R}; (0, \infty))$, $\theta_\pm \in (0, \pi)$: constants$)$

$$\begin{cases} u_t = \frac{u_{xx}}{1+u_x^2} & \text{in } I \times (0, \infty) \\ u_x(\pm b, t) = \pm \tan(\frac{\pi}{2} - \theta_\pm) & \text{for } t > 0 \\ u(x, 0) = u_0(x) & \text{for } x \in \overline{I} \end{cases}$$



✦ Energy structure:

$$E(u(\cdot, t)) := \int \sqrt{1 + u_x^2}\, dx + u(b, t) \cos \theta_+ + u(-b, t) \cos \theta_-$$

decreases

$$u\text{:fix} \Rightarrow \frac{d}{dh} E(u + h) = \cos \theta_+ + \cos \theta_- \begin{cases} < 0, & \theta_+ + \theta_- < \pi \\ = 0, & \theta_+ + \theta_- = \pi \\ > 0, & \theta_+ + \theta_- > \pi \end{cases}$$

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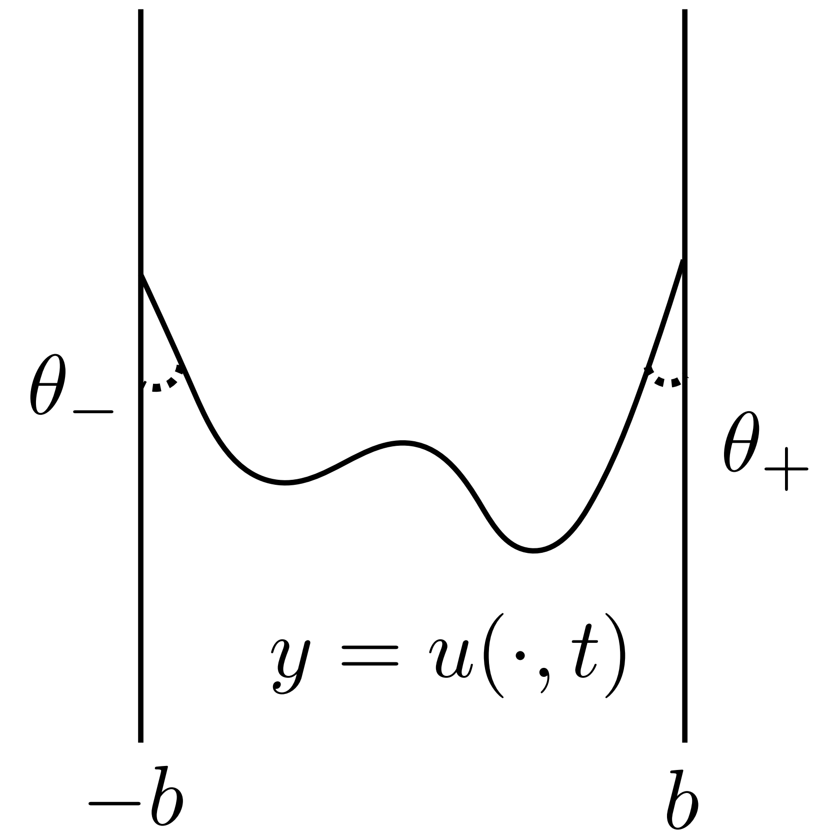

<u>Typical examples of differential equation</u>

- Graph of u is a solution to $V = \kappa$

$$\Rightarrow \quad u_t = \frac{u_{xx}}{1+u_x^2} \quad \left(g(s) = \frac{1}{1+s^2}\right)$$

- Graph of u is a solution to $V = \kappa_\phi$

$$\Rightarrow u_t = \left(\tilde{\phi}_{p_1 p_1}((-u_x, 1)^T)\sqrt{1+u_x^2}\right) u_{xx} \quad \left(\begin{array}{c} \tilde{\phi} : \mathbb{R}^2 \to \mathbb{R} \\ \tilde{\phi}(\lambda\vec{\nu}) := \lambda\phi(\vec{\nu}) \end{array}\right)$$

- Heat equation (which is not geometric flow)

$$u_t = u_{xx} \quad (g(s) = 1)$$

Introduction

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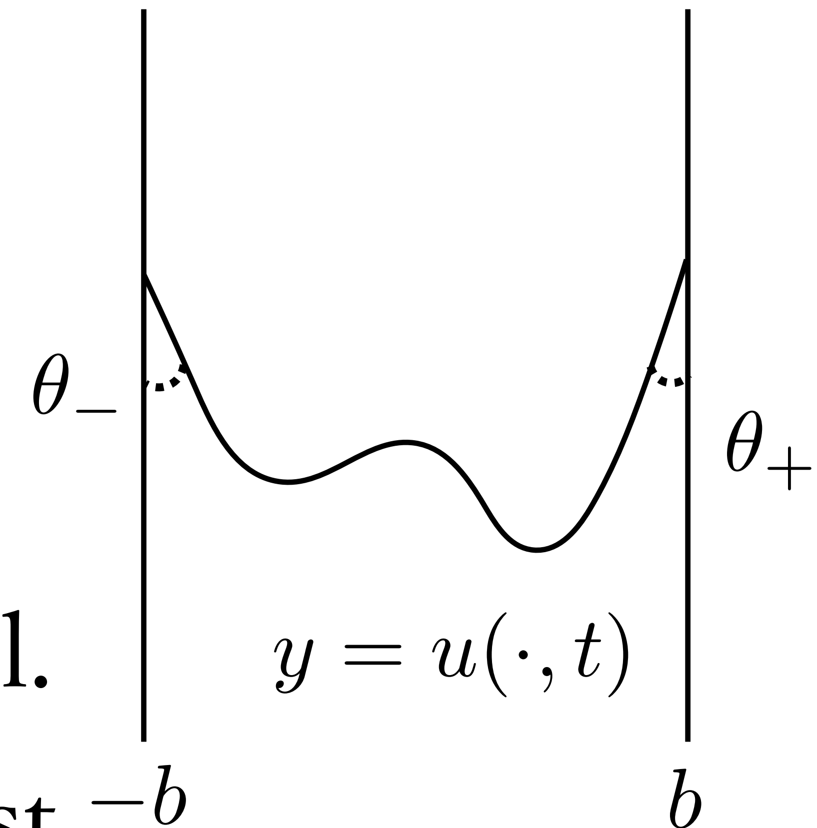

Altschuler-Wu '93:

- $u_0 \in C^\infty(\overline{I}) \Rightarrow$
  $\exists u$: global-in-time (unique) smooth sol.
- As $t \to \infty$, it converges exponentially fast
  to a traveling wave solution $v(x, t) = w(x) + ct$ with

$$\theta_+ + \theta_- \begin{Bmatrix} < \pi \\ = \pi \\ > \pi \end{Bmatrix} \iff c : \text{constant} \begin{Bmatrix} > 0 \\ = 0 \\ < 0 \end{Bmatrix}$$

Introduction

- Capillary rise model $(n=1)$

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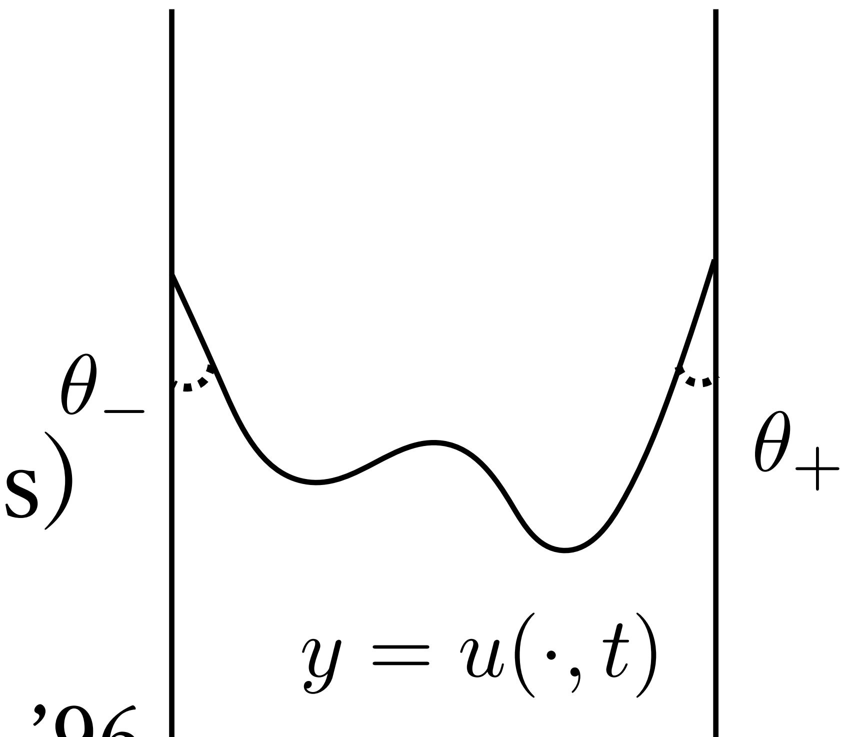

$\longrightarrow$ Generalizations (asymptotic analysis)

- General dimensions and equations

MCF: $(n=2)$ Altschuler-Wu '94, $(n\geq 2)$ Guan '96

General eq.: $(n\geq 1)$ Barles-Da Lio '05, Da Lio '08

- Periodic angle conditions (equation is also generalized)

$$(n=1,\ \theta_\pm(t,u)\in(0,\pi) : \theta_\pm(t+T,u)=\theta_\pm(t,u))$$

Brunovský-Poláčik '92: convergence to periodic sol.

Cai-Lou '11: convergence to periodic traveling wave

Introduction

- Capillary rise model ($n = 1$)

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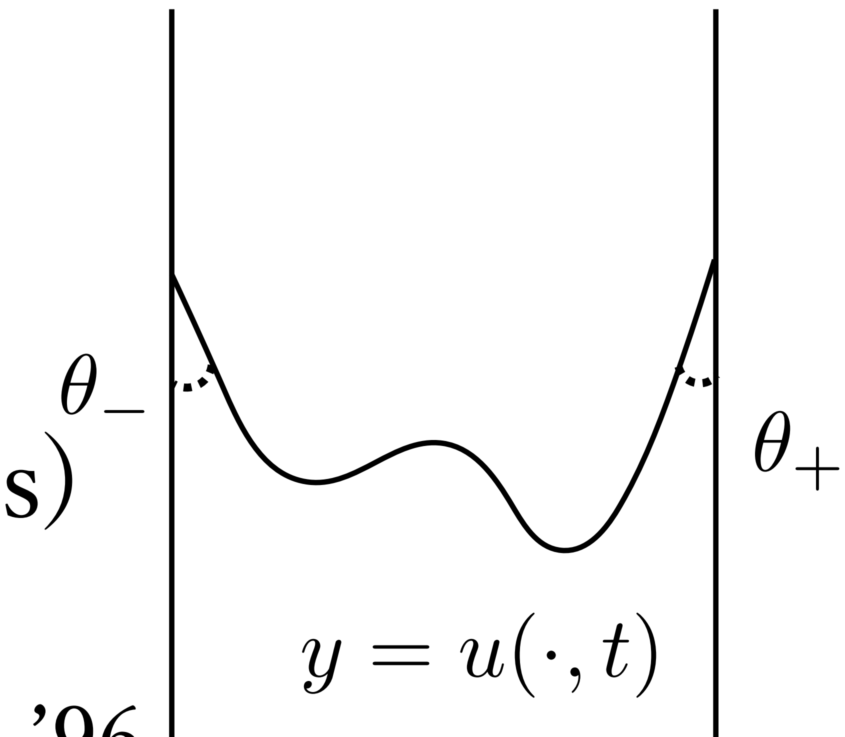

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Question: If $\theta_\pm = 0$, are the solution and traveling wave bounded? (including when finite time)

⟶ Answer: It depends on differential equation (in particular, only on g)

Introduction

cf. Lasry-Lions ('89) ($\Omega \subset \mathbb{R}^n,\ n \geq 1$)

$$-\Delta u + \frac{1}{p}|\nabla u|^p + \varepsilon u = f \ \text{ in } \Omega$$

(Under a suitable condition of f)

- $p > 2 \Rightarrow \exists! u \in C(\overline{\Omega}) \cap C^2(\Omega)$

  solution satisfying $\langle \nabla u, \nu \rangle = \infty$ on $\partial\Omega$

  (graph$u$ tangentially contact to cylinder domain)

- $1 < p \leq 2 \Rightarrow \exists! u \in C^2(\Omega)$

  solution satisfying $u(x) = \infty$ on $\partial\Omega$

Question: If $\theta_{\pm} = 0$, are the solution and traveling wave bounded? (including when finite time)

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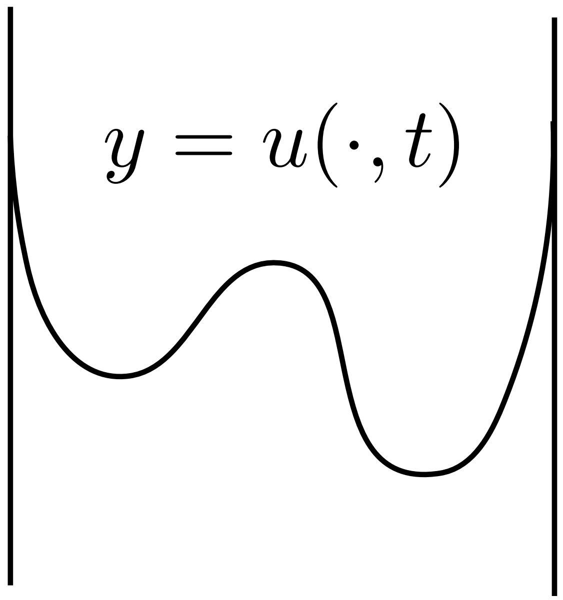
Problem (tangentially contact angle condition)

$I = (-b, b)$: interval



$$\text{(P)}: \begin{cases} u_t = f(g(u_x)u_{xx}) & \text{in } I \times (0, \infty) \\ \lim_{x \to \pm b} u_x(x,t) = \pm\infty & \text{for } t > 0 \\ u(x,0) = u_0(x) & \text{for } x \in \overline{I} \end{cases}$$

(A1) • $f \in C(\mathbb{R})$: increase • $f(0) = 0$ • $f(\pm\infty) = \pm\infty$

(A2) • $g \in C(\mathbb{R}; (0, \infty))$

• $\exists \alpha \in \mathbb{R}, \exists C_\pm > 0$ s.t. $\lim_{s \to \pm\infty} |s|^\alpha g(s) = C_\pm$ $(g(u_x) \approx |u_x|^{-\alpha})$

Typical examples of differential equation

- $f(s) = s \Rightarrow u_t = g(u_x)u_{xx}$ (as above)
- Graph of u is a solution to $V = |\kappa|^{\beta-1}\kappa$ $(\beta > 0)$

$$\Rightarrow u_t = \left| \frac{u_{xx}}{(1+u_x^2)^{\frac{3\beta-1}{2\beta}}} \right|^{\beta-1} \frac{u_{xx}}{(1+u_x^2)^{\frac{3\beta-1}{2\beta}}} \quad \left(\begin{aligned} f(s) &= |s|^{\beta-1}s \\ g(s) &= \frac{1}{(1+s^2)^{\frac{3\beta-1}{2\beta}}} \end{aligned} \right)$$

Boundedness of TW

(P): (Eq) $u_t = f(g(u_x)u_{xx})$ $\left(g(s) \approx |s|^{-\alpha} \text{ as } |s| \to \pm\infty\right)$ (BC) $u_x(\pm b, t) = \pm\infty$

Boundedness of traveling wave

$(v(x,t) = w(x) + ct,\ w \in C^2(I),\ c \in \mathbb{R})$

- $f^{-1}(c) = g(w_x)w_{xx}$: Equation of $w(x)$
- Assume $\alpha > 1,\ w_x(x) \approx (b-x)^\gamma (\gamma < 0)$ around $x = b$

$$\Rightarrow w : \begin{cases} \text{b'dd}, & \gamma > -1 \\ \text{unb'dd}, & \gamma \le -1 \end{cases} \qquad g(w_x)w_{xx} \overset{x \to b}{\to} \begin{cases} \infty, & \gamma > -\frac{1}{\alpha - 1} \\ \text{const}, & \gamma = -\frac{1}{\alpha-1} \\ 0, & \gamma < -\frac{1}{\alpha-1} \end{cases}$$

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$\longrightarrow$ • $\alpha > 2 \Rightarrow w$ is bounded

(u should be bounded and converge to TW as $t \to \infty$)

• $1 < \alpha \le 2 \Rightarrow w$ is unbounded

(u should diverges at least $t = \infty$. How about finite time?)

"Instantaneous" blow-up occurs on the boundary ($\alpha \le 2$)

Main results

(P): (Eq) $u_t = f(g(u_x)u_{xx})$ $\left(g(s) \approx |s|^{-\alpha} \text{ as } |s| \to \pm\infty\right)$ (BC) $u_x(\pm b, t) = \pm\infty$

Thm1 (K.-Liu, '21) $\alpha > 2 \Rightarrow$ unique up to vertical translation

- $\exists w \in C^{\frac{\alpha-2}{\alpha-1}}(\overline{I}) \cap C^2(I)$, $\exists! c > 0$ s.t. $w(x) + ct$: sol. to (P)
- $u_0 \in C(\overline{I}) \Rightarrow \exists!$ viscosity sol. $u \in C(\overline{I} \times [0, \infty))$ to (P)
- $u_0 \in C(\overline{I})$ is convex, g is Lipschitz, f^{-1} is Lipschitz away from $s = 0$

$$\Rightarrow \exists a \in \mathbb{R} \text{ s.t. } \|u(\cdot, t) - (w + ct + a)\|_{L^\infty} \to 0$$

(Existence of sol. can be proved by applying Perron's method)

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Thm2 (K.-Liu, '21)

$\alpha \leq 2 \Rightarrow \nexists$ (b'dd) viscosity sol. $u \in C(\bar{I} \times [0,\infty))$ to (P)

(We can prove
$\exists u \in C(\bar{I} \times [0,\infty))$: sol. to (P) $\Rightarrow u(\pm b, t) = \infty$ $(t > 0)$
by constructing a sequence of sub-solutions)

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Remind $(\beta > 0)$

- Graph type sol. of $V = |\kappa|^{\beta - 1}\kappa \ \left(\alpha = \frac{3\beta - 1}{\beta}\right)$:
  - $\beta \le 1 \Rightarrow$ Thm1 can be applied
  - $\beta > 1 \Rightarrow$ Thm2 can be applied

Future works

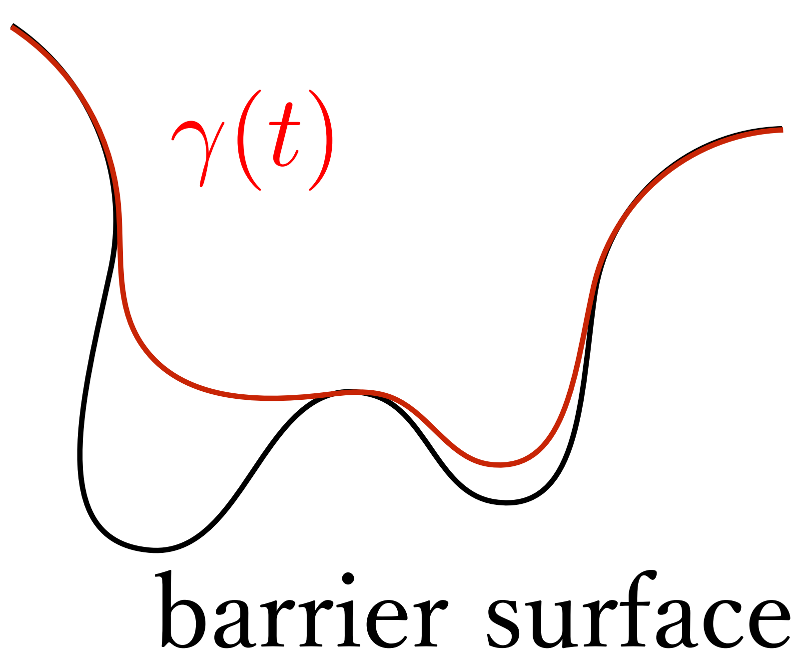
- Higher dimension case
- Singular Dirichlet problem
  - $1 < \alpha \leq 2 \Rightarrow$ Unbounded traveling wave exists

$\longrightarrow$ Unbounded solution should exist and converges to TW

  - $\alpha \leq 1 \Rightarrow$ There is no traveling wave

$\longrightarrow$ (conjecture) Blow-up region may expand from the end points

- Tangentially contact angle condition for geometric flow with general barrier surface



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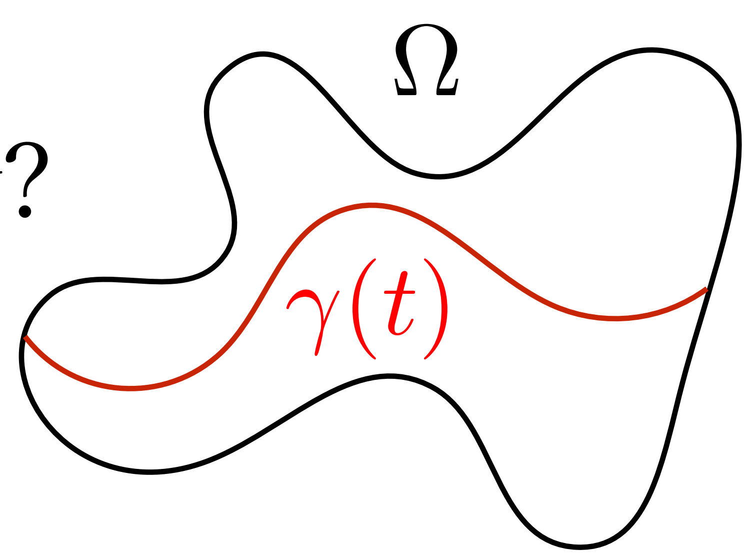
  $\longrightarrow$ (conjecture) Blow-up region may expand from the end points
- Tangentially contact angle condition for geometric flow with general barrier surface
- What happens after moving surface collide with a barrier boundary?



(It should depend on "situations" that topological change occur or moving surface returns to interior)

Thank you for your kind attention!