A convergent algorithm for the interaction of mean curvature flow and surface diffusion

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based on joint work with

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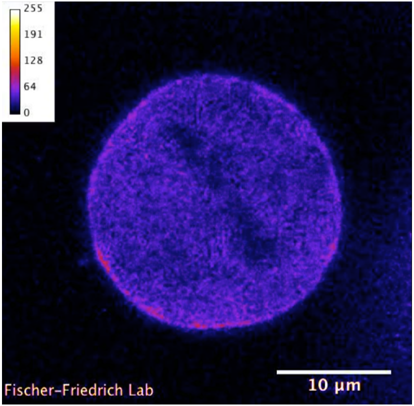
Motivation I. – cell division by contractile ring formation

A bulk–surface model for cell division via surface diffusion of stress generated surface molecules (myosin II), see [Wittwer and Aland (2022)], [Bonati, Wittwer, Aland, and Fischer-Friedrich (2022)].



Experiment by E. Fischer-Friedrich (TU Dresden).

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Motivation II. – A geometric gradient flow

Consider the energy

$$\mathcal{E}(\Gamma[X], u) = \int_{\Gamma[X]} G(u),$$

where

- $\Gamma[X]$ is an evolving surface;
- u is a concentration on the surface $\Gamma[X]$.

The (L^2, H^{-1})-gradient flow of $\mathcal{E}$ yields the *coupled geometric flow*:

$$\begin{aligned} v &= -g(u) H \nu_\Gamma = V \nu_\Gamma, \\ \partial^\bullet u + uVH &= \Delta_{\Gamma[X]} G'(u), \end{aligned}$$

with $g(u) = G(u) - uG'(u)$.

Derivation and analytic theory in [Bürger (2021)].

A generalised coupled problem

A generalised geometric flow interacting with diffusion on $\Gamma[X]$:

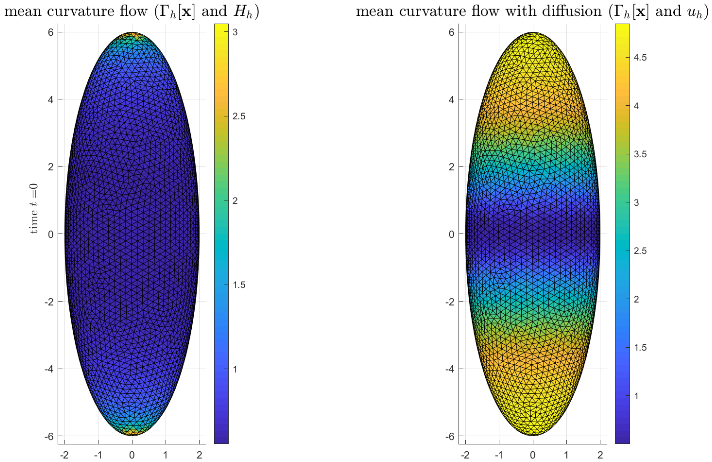
$$v = V\nu, \quad \text{with} \quad V = -F(u, H),$$
$$\partial^\bullet u + u\left(\nabla_{\Gamma[X]} \cdot v\right) = \nabla_{\Gamma[X]} \cdot \left(D(u)\nabla_{\Gamma[X]} u\right),$$

where $F(\cdot, \cdot)$ is a suitable function.

Includes many classical flows:

$F(u, H) = \mp H^{\pm}$,	inverse / mean curvature flow,
$F(u, H) = \mp H^{\pm\alpha}$,	powers of inverse / mean curvature ($\alpha > 0$),
$F(u, H) = -H + g(u)$,	additive forcing,
$F(u, H) = -g(u)H$,	[Bürger (2021)],
etc.	

Mean curvature flow and the coupled geometric flow



Mean curvature flow and the coupled geometric flow

Outline

- Notations
- Two algorithms for mean curvature flow
- Coupled system for the coupled geometric flow
- Evolving surface finite elements and matrix–vector formalism
- Stability analysis — energy estimates
- Convergence
- Numerical experiments

Notations

Evolving surfaces

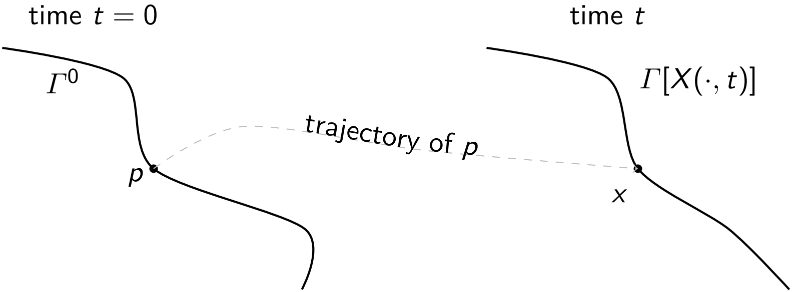
Let $\Gamma(t) \subset \mathbb{R}^3$ be a closed surface parametrised by X over an initial surface Γ^0:

$$\Gamma[X] = \Gamma[X(\cdot,t)] = \{X(p,t) \,:\, p \in \Gamma^0\}.$$

Surface velocity v satisfies, in $x(t) = X(p,t)$, by

$$\partial_t x(t) = \partial_t X(p,t) = v(X(p,t),t) = v(x(t),t).$$

The surface $\Gamma[X(\cdot,t)]$ is a collection of points x, where $x = X(p,t)$ is obtained by solving the above ODE from 0 to t for a fixed p.



Differential operators on $\Gamma[X]$

- Outward normal vector: $\nu = \nu_{\Gamma[X]}$
- Material derivative: $\partial^\bullet u(\cdot, t) = \frac{\mathrm{d}}{\mathrm{d}t}\big(u(X(\cdot, t), t)\big)$
- Tangential gradient: $\nabla_{\Gamma[X]} u = \nabla \overline{u} - (\nabla \overline{u} \cdot \nu)\nu : \Gamma \to \mathbb{R}^3$
- Laplace–Beltrami operator: $\Delta_{\Gamma[X]} u = \nabla_{\Gamma[X]} \cdot \nabla_{\Gamma[X]} u$

  (for $u : \Gamma \to \mathbb{R}$, on a regular surface $\Gamma \subset \mathbb{R}^3$)

Geometric quantities and mean curvature H

- extended Weingarten map (3×3 symmetric matrix)

$$A(x) = \nabla_\Gamma \nu(x)$$

contains geometric informations

- with eigenvalues: κ_1 and κ_2, the principal curvatures,
  and 0 (with eigenvector ν)

- they define

$$\text{mean curvature} \qquad H = \mathrm{tr}(A) = \kappa_1 + \kappa_2,$$
$$\text{and} \qquad |A|^2 = \|A\|_F^2 = \kappa_1^2 + \kappa_2^2.$$

Two algorithms for mean curvature flow

MCF and Dziuk's algorithm

A regular surface $\Gamma[X]$ moving under mean curvature flow satisfies:

$$\partial_t X = v \circ X,$$
$$v = -H\nu.$$

Heat-like equation, using that on any Γ: $-H\nu = \Delta_\Gamma x_\Gamma$ (where $x_\Gamma = \mathrm{id}_\Gamma$):

$$\partial_t X(p,t) = \Delta_{\Gamma[X]} x_{\Gamma[X]}.$$

[Dziuk (1990)]

Simple and elegant algorithm;
computes all geometry from surface.

A coupled system for mean curvature flow

Inspired by [Huisken (1984)], consider the coupled system:

$$v = -H\nu,$$

$$\partial^\bullet \nu = \Delta_{\Gamma[X]}\nu + |A|^2\,\nu,$$
$$\partial^\bullet H = \Delta_{\Gamma[X]}H + |A|^2 H,$$

$$\partial_t X = v \circ X.$$

First convergence proof for MCF in [K., Li, and Lubich (2019)]:
optimal-order H^1 norm error estimates (for evolving surface FEM of order $k \geq 2$ and BDF of order 2 to 5).

Leads to a less simple, but natural algorithm;
computes all geometry from evolution equations.

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Coupled system for the interaction of mean curvature flow and diffusion

Interaction of mean curvature flow and diffusion

Instead of mean curvature flow

$$v = (-H)\nu_\Gamma,$$

consider now the generalised mean curvature flow

$$v = V\nu_\Gamma \quad \text{with} \quad V = -F(u, H),$$

with a given function F.

The real question is:
How robust is our approach from [KLL (2019)]?

Brief answer: Very!!

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Not-so-brief answer – recap mean curvature flow

Following [Huisken (1984)], for a regular surface $\Gamma[X]$ the identities hold:

$$\nabla_\Gamma H = \Delta_\Gamma \nu + |A|^2 \nu, \qquad \text{and} \tag{1}$$

$$\partial^\bullet \nu = -\nabla_\Gamma V, \tag{2}$$

$$\partial^\bullet H = -\Delta_\Gamma V - |A|^2 V. \tag{3}$$

$$V = -H. \tag{4a}$$

For example:

$$\begin{aligned} \partial^\bullet \nu &\overset{(2)}{=} -\nabla_\Gamma V \\ &\overset{(4a)}{=} -\nabla_\Gamma(-H) \\ &\overset{(1)}{=} \Delta_\Gamma \nu + |A|^2 \nu. \end{aligned}$$

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$$V = F(u, H) = -g(u)H. \tag{4b}$$

For example

$$\begin{aligned} \partial^\bullet \nu &\overset{(2)}{=} -\nabla_\Gamma V \\ &\overset{(4b)}{=} -\nabla_\Gamma(-g(u)H) \\ &= g(u)\nabla_\Gamma H + H\nabla_\Gamma(g(u)) \\ &\overset{(1)}{=} g(u)\Big(\Delta_\Gamma \nu + |A|^2\nu\Big) + H\nabla_\Gamma(g(u)). \qquad (/g(u) > 0) \end{aligned}$$

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A coupled system for the interaction of MCF and diffusion

For $V = -F(u, H)$ with inverse $H = -K(u, V)$ (for fixed u).
Coupled system with fundamental variables X, v, ν, V and u:

$$\partial_t X = v \circ X,$$

$$v = V\nu,$$

$$\partial_2 K \, \partial^\bullet \nu = \Delta_{\Gamma[X]}\nu + |A|^2 \nu + \partial_1 K \, \nabla_{\Gamma[X]} u,$$
$$\partial_2 K \, \partial^\bullet V = \Delta_{\Gamma[X]} V + |A|^2 V - \partial_1 K \, \partial^\bullet u,$$

$$\partial^\bullet u + u \left(\nabla_{\Gamma[X]} \cdot v\right) = \nabla_{\Gamma[X]} \cdot \left(D(u) \nabla_{\Gamma[X]} u\right).$$

Still a natural algorithm,
which comes with a convergence analysis.

Evolving surface finite elements and matrix–vector formulation

Semi-discrete problem

Evolving surface FEM [Dziuk and Elliott], [Demlow (2009)];
nodal values $z_h \rightsquigarrow \mathbf{z}$ (for all finite element functions).

$$\begin{aligned} \partial_t X_h &= v_h \circ X_h, \\ \text{with} \qquad v_h &= \widetilde{I}_h(V_h \nu_h), \end{aligned}$$

for $w_h = (\nu_h, V_h)$

$$\begin{aligned} \int_{\Gamma_h[\mathbf{x}]} \partial_2 K_h \, \partial_h^\bullet w_h \cdot \varphi_h^w + \int_{\Gamma_h[\mathbf{x}]} \nabla_{\Gamma_h[\mathbf{x}]} w_h \cdot \nabla_{\Gamma_h[\mathbf{x}]} \varphi_h^w \\ = \int_{\Gamma_h[\mathbf{x}]} |A_h|^2 w_h \cdot \varphi_h^w + \int_{\Gamma_h[\mathbf{x}]} f(\partial_1 K_h, w_h, u_h; \partial_h^\bullet u_h) \cdot \varphi_h^w, \end{aligned}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg(\int_{\Gamma_h[\mathbf{x}]} u_h \, \varphi_h^u \bigg) + \int_{\Gamma_h[\mathbf{x}]} D(u_h) \, \nabla_{\Gamma_h[\mathbf{x}]} u_h \cdot \nabla_{\Gamma_h[\mathbf{x}]} \varphi_h^u = \int_{\Gamma_h[\mathbf{x}]} u_h \, \partial_h^\bullet \varphi_h^u,$$

Matrix–vector formulation

Upon setting $\mathbf{w} = \big(\mathbf{n}, \mathbf{V}\big)^T \in \mathbb{R}^{4N}$, the semi-discrete problem is equivalent to the following differential algebraic system:

$$
\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{v}, \\
\mathbf{v} &= \mathbf{V} \bullet \mathbf{n}, \\
\mathbf{M}(\mathbf{x}, \mathbf{u}, \mathbf{w})\dot{\mathbf{w}} + \mathbf{A}(\mathbf{x})\mathbf{w} &= \mathbf{f}(\mathbf{x}, \mathbf{w}, \mathbf{u}; \dot{\mathbf{u}}), \\
\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathbf{M}(\mathbf{x})\mathbf{u}\Big) + \mathbf{A}(\mathbf{x}, \mathbf{u})\mathbf{u} &= 0.
\end{aligned}
$$

Used for computation and analysis.

Stability analysis:
relating surfaces and energy estimates

Stability via energy estimates

A key issue is to compare different quantities on different meshes. For this we need pointwise $W^{1,\infty}$ norm bound on the position errors.

(i) Obtain pointwise H^1 norm stability estimates over $[0, T^*]$,
using **energy estimates**,
testing with time derivatives of the errors
(fully discrete: first done in [KLL (2019)]).

(ii) Using an inverse estimate to establish bounds in the $W^{1,\infty}$ norm.

(iii) Prove that in fact $T^* = T$.

Similarly to [K., Li, and Lubich (2019,2020)]
and [Binz and K. (2021)]

Semi-discrete error estimates

Semi-discrete convergence estimates

Consider the semi-discretisation of the coupled system for the interaction of mean curvature flow and diffusion using ESFEM of polynomial degree $k \geq 2$.

Let the solutions (X, v, ν, V, u) be sufficiently smooth.

Then for sufficiently small h the following estimates hold for $0 \leq t \leq T$:

$$
\begin{aligned}
\|(x_h(\cdot, t_n))^L - \mathrm{id}_{\Gamma(t_n)}\|_{H^1(\Gamma(t_n))^3} &\leq Ch^k, \\
\|(v_h(\cdot, t_n))^L - v(\cdot, t_n)\|_{H^1(\Gamma(t_n))^3} &\leq Ch^k, \\
\|(\nu_h(\cdot, t_n))^L - \nu(\cdot, t_n)\|_{H^1(\Gamma(t_n))^3} &\leq Ch^k, \\
\|(V_h(\cdot, t_n))^L - V(\cdot, t_n)\|_{H^1(\Gamma(t_n))} &\leq Ch^k, \\
\|(u_h(\cdot, t_n))^L - u(\cdot, t_n)\|_{H^1(\Gamma(t_n))} &\leq Ch^k.
\end{aligned}
$$

The constant $C > 0$ is independent of h, but depends on the solution and on T.

[Elliott, Garcke, and K. (2022)]

Numerical experiments

Properties of MCF and the gradient flow

(i) Conservation of mean-convexity: **[both]**

$$\text{if} \quad H(\cdot,0) \geq 0, \quad \text{then} \quad H(\cdot,t) > 0, \ \forall t.$$

(i) Loss of convexity: **[MCF preserves]**

$$\text{if} \quad \Gamma^0 \text{ is convex}, \quad \text{then} \quad \Gamma[X(\cdot,t)] \text{ is not necessarily convex.}$$

(iii) Formation of self-intersections are possible. **[not for MCF]**

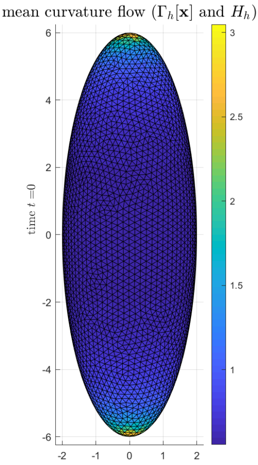
(iv) Concentration properties:

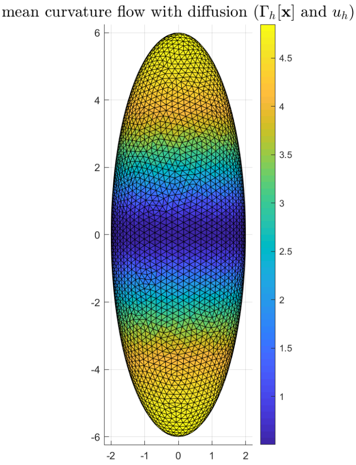
$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Gamma[X]} u = 0, \qquad u(\cdot,0) \geq 0 \ \Rightarrow \ u(\cdot,t) \geq 0, \ \forall t, \qquad \min\{u\} \nearrow .$$

[Huisken (1984)]
[Bürger (2021)]

All observable in numerical experiments.

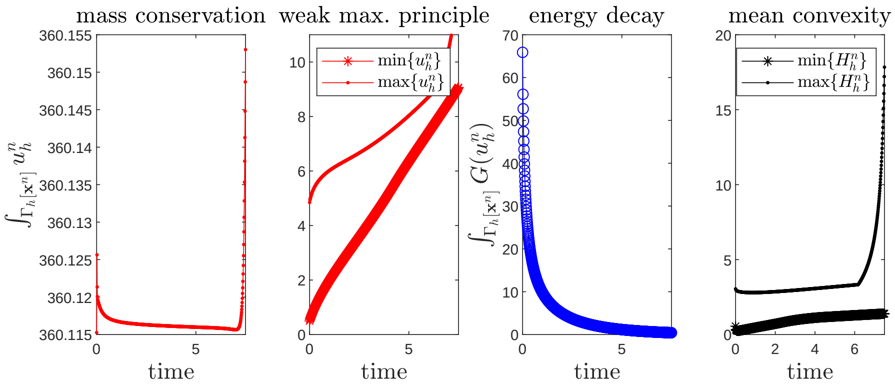
Loss of convexity, while preserving mean convexity



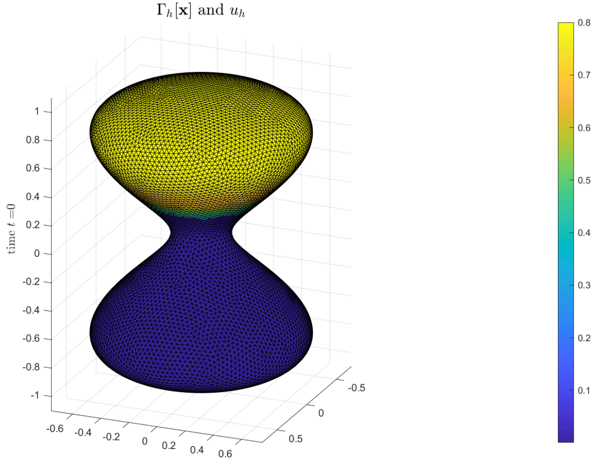


Loss of convexity, while preserving mean convexity

Qualitative properties of the fully discrete solution



Slow diffusion through a tight neck



cf. [Ecker (2008)]

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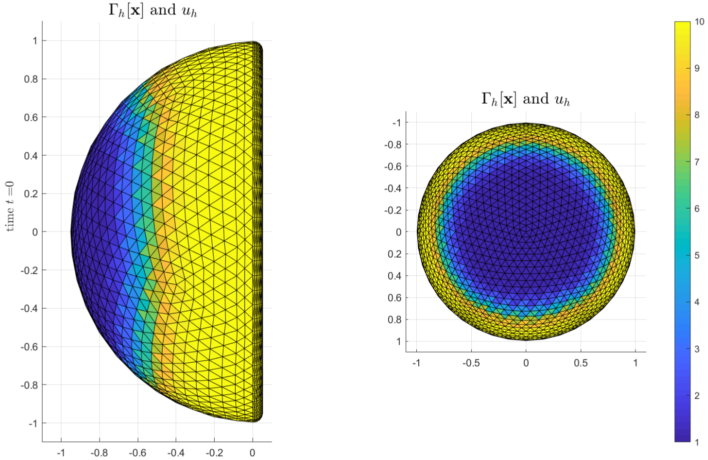
Self-intersection



Thank you for your attention!

Self-intersection

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A key observations

We use dynamic variables to determine
the geometric quantities in the surface velocity $v_h \approx V_h \nu_h$.

	exact solution	approximation	geometry
surface:	$X(\cdot, t) : \Gamma^0 \to \mathbb{R}^3$	$X_h(\cdot, t) : \Gamma_h^0 \to \mathbb{R}^3$ (collected into $\mathbf{x}(t)$)	
velocity:	$v : \Gamma[X] \to \mathbb{R}^3$	$v_h : \Gamma_h[\mathbf{x}] \to \mathbb{R}^3$	
surface normal:	$\nu : \Gamma[X] \to \mathbb{S}^3$	$\nu_h : \Gamma_h[\mathbf{x}] \to \mathbb{R}^3$	$\neq \nu_{\Gamma_h[\mathbf{x}]} \in \mathbb{S}^3$
normal velocity:	$V : \Gamma[X] \to \mathbb{R}$	$V_h : \Gamma_h[\mathbf{x}] \to \mathbb{R}$	$\neq V_{\Gamma_h[\mathbf{x}]}$

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Defects and comparing surfaces

Deriving error equations – identifying problems

Numerical scheme:

$$
\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{v}, \\
\mathbf{v} &= \mathbf{V} \bullet \mathbf{n}, \\
\mathbf{M}(\mathbf{x}, \mathbf{u}, \mathbf{w})\dot{\mathbf{w}} + \mathbf{A}(\mathbf{x})\mathbf{w} &= \mathbf{f}(\mathbf{x}, \mathbf{w}, \mathbf{u}; \dot{\mathbf{u}}), \\
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Exact solutions in the method:

$$
\begin{aligned}
\dot{\mathbf{x}}^* &= \mathbf{v}, \\
\mathbf{v}^* &= \mathbf{V}^* \bullet \mathbf{n}^* + \mathbf{d}_{\mathbf{v}}, \\
\mathbf{M}(\mathbf{x}^*, \mathbf{u}^*, \mathbf{w}^*)\dot{\mathbf{w}}^* + \mathbf{A}(\mathbf{x}^*)\mathbf{w}^* &= \mathbf{f}(\mathbf{x}^*, \mathbf{w}^*, \mathbf{u}^*; \dot{\mathbf{u}}^*) + \mathbf{M}(\mathbf{x}^*)\mathbf{d}_{\mathbf{w}}, \\
\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathbf{M}(\mathbf{x}^*)\mathbf{u}^*\Big) + \mathbf{A}(\mathbf{x}^*, \mathbf{u}^*)\mathbf{u}^* &= \mathbf{M}(\mathbf{x}^*)\mathbf{d}_{\mathbf{u}}.
\end{aligned}
$$

Error equations and stability

We aim to prove stability:

$$\|\text{errors}(\cdot,t)\|^2 \leq \|\text{errors}(\cdot,0)\|^2 + \int_0^t \|\text{defects}(\cdot,s)\|^2 \text{d}s.$$

As discussed, the error equations contain some problematic terms:

$$\mathbf{M}(\mathbf{x}) - \mathbf{M}(\mathbf{x}^*) \quad \text{and} \quad \mathbf{A}(\mathbf{x}) - \mathbf{A}(\mathbf{x}^*),$$

$$\mathbf{M}(\mathbf{x},\mathbf{u},\mathbf{w})\dot{\mathbf{w}} - \mathbf{M}(\mathbf{x}^*,\mathbf{u}^*,\mathbf{w}^*)\dot{\mathbf{w}}^*,$$

$$\mathbf{f}(\mathbf{x},\mathbf{w},\mathbf{u};\dot{\mathbf{u}}) - \mathbf{f}(\mathbf{x}^*,\mathbf{w}^*,\mathbf{u}^*;\dot{\mathbf{u}}^*)$$

($\mathbf{f}$ is only locally Lipschitz).

Suitable comparisons subject to the condition $\|e_x\|_{W^{1,\infty}(\Gamma_h[\mathbf{x}_*])} \leq \frac{1}{4}$**.**

Relating different surfaces – I.

In order to study errors,
we need to compare quantities on different surfaces.

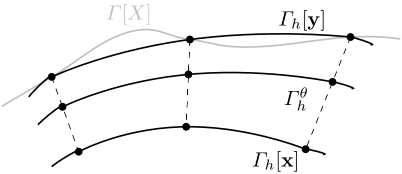
Let $\mathbf{x} \in \mathbb{R}^{3N}$ and $\mathbf{x}_* \in \mathbb{R}^{3N}$ be two vectors which define the surfaces $\Gamma_h[\mathbf{x}]$ and $\Gamma_h[\mathbf{x}_*]$.

Intermediate surfaces for $\theta \in [0,1]$:

$$\mathbf{e}_{\mathbf{x}} = \mathbf{x} - \mathbf{x}_* \rightsquigarrow \Gamma_h^\theta = \Gamma_h[\mathbf{x}_* + \theta \mathbf{e}_{\mathbf{x}}],$$

and the corresponding errors:

$$e_x^\theta \text{ on } \Gamma_h[\mathbf{x}_* + \theta \mathbf{e}_{\mathbf{x}}].$$



(Lift operation: u_h^L)

Relating different surfaces – II.

Key tools are: technical lemmas, and techniques, which relate different evolving surfaces with one another.

For example:

$$\|\mathbf{w}\|_{\mathbf{M}(\mathbf{x}_*+\theta\mathbf{e}_\mathbf{x})} \leq c\,\|\mathbf{w}\|_{\mathbf{M}(\mathbf{x}_*)},$$
$$\|\nabla_{\Gamma_h^\theta} w_h^\theta\|_{L^p(\Gamma_h^\theta)} \leq c_p\,\|\nabla_{\Gamma_h^0} w_h^0\|_{L^p(\Gamma_h^0)},$$
$$\mathbf{w}^T(\mathbf{M}(\mathbf{x}) - \mathbf{M}(\mathbf{x}_*))\mathbf{z} \leq c\|\mathbf{w}\|_{\mathbf{M}(\mathbf{x}_*)}\|\mathbf{z}\|_{\mathbf{M}(\mathbf{x}_*)},$$

etc.

[K., Li, Lubich and Power (2017)]
[K., Li, and Lubich (2019)]
[Elliott, Garcke and K. (2022)]

Under the important condition on $\mathbf{e}_\mathbf{x}$: $\|e_x^0\|_{W^{1,\infty}(\Gamma_h[\mathbf{x}_*])} \leq \frac{1}{4}$.

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Another typical lemma

Assume that $\|\nabla_{\Gamma_h[y]} e_h^0\|_{L^\infty(\Gamma_h[\mathbf{y}])} \leq \frac{1}{4}$:

$$\mathbf{w}^T \big(\mathbf{M}(\mathbf{x},\mathbf{u},\mathbf{V}) - \mathbf{M}(\mathbf{y},\mathbf{u},\mathbf{V})\big)\mathbf{z} \leq C\, \|\nabla_{\Gamma_h[y]} e_h^0\|_{L^\infty(\Gamma_h[\mathbf{y}])}\, \|\mathbf{w}\|_{\mathbf{M}(\mathbf{y})}\, \|\mathbf{z}\|_{\mathbf{M}(\mathbf{y})},$$

and

$$\begin{aligned} &\mathbf{w}^T \big(\mathbf{M}(\mathbf{x},\mathbf{u},\mathbf{V}) - \mathbf{M}(\mathbf{x},\mathbf{u}^*,\mathbf{V}^*)\big)\mathbf{z} \\ &\qquad \leq C\, \big(\|u_h - u_h^*\|_{L^\infty(\Gamma_h[\mathbf{y}])} + \|V_h - V_h^*\|_{L^\infty(\Gamma_h[\mathbf{y}])}\big)\, \|\mathbf{w}\|_{\mathbf{M}(\mathbf{x})}\, \|\mathbf{z}\|_{\mathbf{M}(\mathbf{x})}, \end{aligned}$$

The constant $C > 0$ is independent of h and t.

[Elliott, Garcke and K. (2022)].

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Main idea of fully discrete stability analysis

[KLL (2019)]

Illustrate using a simple case

Consider the weak form of the heat equation (with appr. b.c.):

$$(\dot u(t), \varphi) + (\nabla u(t), \nabla \varphi) = (f(t), \varphi), \\ u(0) = u_0.$$

Energy estimates, let $\varphi = u$ and $\varphi = \dot u$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \leq c\|f\|_*^2, \tag{a}$$

$$\|\dot u\|_{L^2}^2 + \frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_{L^2}^2 \leq c|f|^2, \tag{b}$$

then integrate in time.

“Repeat” for time discrete error equation, testing with $\dot{\mathrm{e}}^n$.

G-stability of [Dahlquist (1978)] and the multiplier techniques of [Nevanlinna and Odeh (1981)]

Dahlquist and Nevanlinna & Odeh

Dahlquist's G-stability theory: Let $\delta(\zeta)$ and $\mu(\zeta)$ be polynomials of degree at most k. If

$$\mathrm{Re}\,\frac{\delta(\zeta)}{\mu(\zeta)} > 0, \qquad \text{for} \quad |\zeta| < 1,$$

then there exists $G = (g_{ij}) \in \mathbb{R}^{k\times k}$ s.p.d. such that for all $v_0, \dots, v_k \in \mathbb{R}^N$

$$\Big\langle \sum_{i=0}^{k} \delta_i v_{k-i} \Big| \sum_{i=0}^{k} \mu_i v_{k-i} \Big\rangle \geq \sum_{i,j=1}^{k} g_{ij} \langle v_i \mid v_j \rangle - \sum_{i,j=1}^{k} g_{ij} \langle v_{i-1} \mid v_{j-1} \rangle.$$

Multiplier technique of Nevanlinna & Odeh: If $k \leq 5$, then there exists $0 \leq \eta < 1$ such that for $\delta(\zeta) = \sum_{\ell=1}^{k} \frac{1}{\ell}(1-\zeta)^\ell$,

$$\mathrm{Re}\,\frac{\delta(\zeta)}{1-\eta\zeta} > 0, \qquad \text{for} \quad |\zeta| < 1.$$

The smallest possible values of η is found to be
$\eta = 0,\ 0,\ 0.0836,\ 0.2878,\ 0.8160$ for $k = 1, 2, \dots, 5$, respectively.

Energy estimates for BDF methods

Using *G*-stability of [Dahlquist (1978)] and the multiplier techniques of [Nevanlinna and Odeh (1981)]:

Testing with multiplier $u^n - \eta u^{n-1}$ (A-stable: $\eta = 0$, $A(\alpha)$-stable: $0 < \eta < 1$):

$$(\dot{u}^n, u^n - \eta u^{n-1}) + (Au^n, u^n - \eta u^{n-1}) = (f^n, u^n - \eta u^{n-1}). \qquad \text{(a)}$$

for PDEs: [Lubich, Mansour and Venkataraman (2013)], [Akrivis and Lubich (2015)], ...

Testing with $\dot{u}^n$:

$$(\dot{u}^n, \dot{u}^n) + (Au^n, \dot{u}^n) = (f^n, \dot{u}^n). \qquad \text{(b)}$$

Where is the multiplier?

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Subtract the equations at time t_{n-1} from at time t_n, and test with $\dot{u}^n$:

$$(\dot{u}^n - \eta \dot{u}^{n-1}, \dot{u}^n) + (Au^n - \eta Au^{n-1}, \dot{u}^n) = (f^n - \eta f^{n-1}, \dot{u}^n). \qquad \text{(b)}$$

Which yields a pointwise stability estimate in the strong norm.

Sketch of stability proof

Using G-stability of [Dahlquist (1978)] and the multiplier techniques of [Nevanlinna and Odeh (1981)] for the second term.

$$(\dot{e}^n - \eta \dot{e}^{n-1}, \dot{e}^n) + (A e^n - \eta A e^{n-1}, \dot{e}^n) = (d^n - \eta d^{n-1}, \dot{e}^n)$$

back

Sketch of stability proof

Using G-stability of [Dahlquist (1978)] and the multiplier techniques of [Nevanlinna and Odeh (1981)] for the second term.

$$
\begin{aligned}
&\Big(1-\frac{\eta}{2}\Big)|\dot e^n|^2 - \frac{\eta}{2}|\dot e^{n-1}|^2 \\
&\quad + \tfrac{1}{\tau}\sum_{i,j=1}^{q} g_{ij}(Ae^{n-q+i}, e^{n-q+j}) - \tfrac{1}{\tau}\sum_{i,j=1}^{q} g_{ij}(Ae^{n-1-q+i}, e^{n-1-q+j}) \\
&\leq (\dot e^n - \eta \dot e^{n-1}, \dot e^n) + (Ae^n - \eta Ae^{n-1}, \dot e^n) = (d^n - \eta d^{n-1}, \dot e^n) \\
&\leq \varepsilon |\dot e^n|^2 + c\big(|d^n|^2 + |d^{n-1}|^2\big)
\end{aligned}
$$

(multiply by τ and sum up; Gronwall)

back